2 month Internship

Statistical Distributions on the Sphere and their Applications to Light Polarization

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Laboratory Overview

[1] The ICube Laboratory is a prominent research center specializing in engineering science, computer science, and imaging. Established in 2013, the laboratory is a collaborative effort involving researchers from the University of Strasbourg, the CNRS (French National Center for Scientific Research), ENGEES, and INSA Strasbourg. The lab's research is unified by its focus on imaging, which serves as a central theme across its various disciplines.
With a team of approximately 650 members, ICube is a leading research entity in Strasbourg, driving advancements particularly in the fields of biomedical engineering and sustainable development.
As a member of the Carnot Telecom and Société numérique institute, ICube benefits from a strong partnership with Telecom Physique Strasbourg, further enhancing its research capabilities and industry connections.

During my internship, I was part of the Imaging, Robotics, Remote Sensing, and Biomedical Department (D-IRTS), under the leadership of department head Edouard Laroche. Specifically, I worked with the Remote Sensing, Radiometry, and Optical Imaging team (TRIO), led by team head Jihad Zallat, with Pierre Grussenmeyer serving as deputy. I was directly supervised by Jihad Zallat and Jean Rehbinder. Jean Rehbinder is an Engineer/Technician/Administrator (ITA) assigned to both the TRIO and GAIA teams.

1 Context

Polarization of light is one of its fundamental characteristics, with applications in biomedical imaging, remote sensing, and the characterization of components and materials. In this work, we aim to understand and model polarization distributions and their applications for analyzing data from various polarimeters.
The coherence matrix and the Poincaré sphere provide tools to represent and manipulate partial polarization states and to introduce the degree of polarization in a mathematically rigorous and visually intuitive manner.
Additionally, statistical distributions on the sphere, notably the von Mises-Fisher distributions, offer a powerful framework for modeling directions and orientations in a three-dimensional space and, by extension, polarization states on the Poincaré sphere.

2 Introduction

An electric field propagating along the z axis is described by the vector $E = (E_x \exp(i\omega t), E_y \exp(i\omega t), 0)^T$ where E_x and E_y are complex amplitudes.

2.1 Stokes vectors

[14] Over time, the total polarization state describes an ellipse.

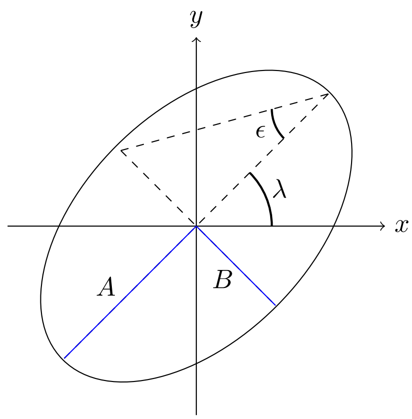

Figure 1: Definition of the azimuth λ, ϵ and the semi major A and minor B axis of the ellipse.

λ is call the *azimuth*, it's the angle between the major axis of the ellipse and the x axis. $(-90 \leq \lambda \leq 90)$. ϵ is defined as $\tan(\epsilon) = \frac{B}{A}$.

The *Stokes vectors* S describes the polarization state of electromagnetic radiation with four values : $S = (I, Q, U, V)^T$.

$$\begin{cases} I &= E_x E_x^* + E_y E_y^* \\ Q &= E_x E_x^* - E_y E_y^* \\ U &= E_x E_y^* + E_x^* E_y \\ V &= i(E_x^* E_y - E_x E_y^*) \end{cases}$$

With the polarization ellipse parameters :

$$\begin{cases} I &= A^2 + B^2 \\ Q &= (A^2 - B^2)\cos(2\lambda) \\ U &= (A^2 - B^2)\sin(2\lambda) \\ V &= 2ABh, \ h = \operatorname{sgn}(V) \end{cases}$$

The degree of polarization (DOP) is define : $DOP(S) = \frac{\sqrt{Q^2+U^2+V^2}}{I} \leq 1$, if $DOP(S) = 1$ the light is totally polarize.

Remark 1. If we normalize S by I : $S_{norm} = \left(1, \frac{Q}{I}, \frac{U}{I}, \frac{V}{I}\right)^T$, the expression of the DOP is the norm of the vector of the last three coordinates of S.

2.2 The Poincaré sphere

Each point on the Poincaré sphere represents the polarization state of light. If a point is on the surface of the sphere, it indicates that the light is totally polarized (the DOP equals 1). Points inside the sphere represent partially polarized light (the DOP is less than 1 but greater than 0).

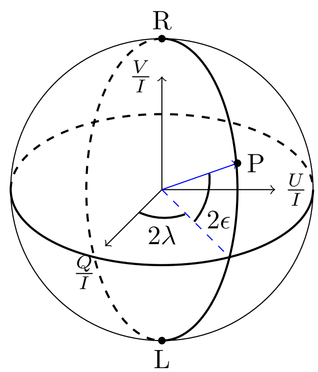

Figure 2: Poincaré sphere with the definition of the direction vectors, 2λ and 2ϵ for a random point P on the sphere.

$$\begin{cases} \frac{Q}{I} &= \cos(2\lambda)\cos(2\epsilon) \\ \frac{U}{I} &= \sin(2\lambda)\cos(2\epsilon) \\ \frac{V}{I} &= \sin(2\epsilon) \end{cases}$$

[7]

Proof. We have : $\tan(\epsilon) = \frac{B}{A}$, so $\tan(2\epsilon) = \frac{2\tan(\epsilon)}{1-\tan^2(\epsilon)} = \frac{2\frac{B}{A}}{1-\frac{B^2}{A^2}} = \frac{2AB}{A^2-B^2}$.

$\cos(2\epsilon) = \frac{1-\tan^2(\epsilon)}{1+\tan^2(\epsilon)} = \frac{1-\frac{B^2}{A^2}}{1+\frac{B^2}{A^2}} = \frac{A^2-B^2}{A^2+B^2}$

$\sin(2\epsilon) = \tan(2\epsilon)\cos(2\epsilon) = \frac{2AB}{A^2-B^2}\frac{A^2-B^2}{A^2+B^2} = \frac{2AB}{A^2+B^2}$

- $\frac{Q}{I} = \cos(2\lambda)\frac{A^2-B^2}{A^2+B^2} = \cos(2\lambda)\cos(2\epsilon)$
- $\frac{U}{I} = \sin(2\lambda)\frac{A^2-B^2}{A^2+B^2} = \sin(2\lambda)\cos(2\epsilon)$
- $\frac{V}{I} = \frac{2AB}{A^2+B^2} = \sin(2\epsilon)$

□

2.3 Mueller matrices

[5]

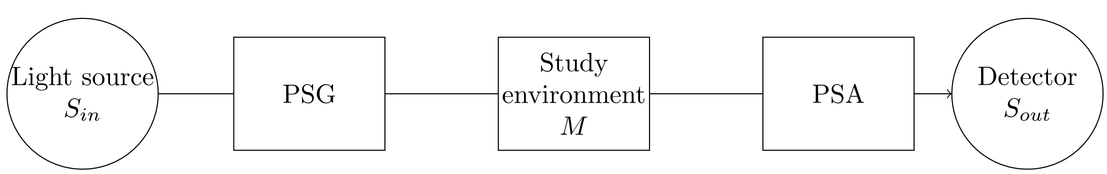

Figure 3: Schema of a Mueller polarimetric imaging system.
PSG = Polarization State Generator and PSA = Polarization State Analyzer.

A Mueller matrix M is define : $S_{out} = MS_{in}$, we have : S_{out} the Storkes vector of the outgoing light and S_{in} the Storkes vector of the incident light.

For describe a system of polarization, we use the Mueller matrices or the Jones matrices with this correspondence :

non-polarized	Mueller matrix
partially polarized	Mueller matrix
totally polarized	Mueller matrix, Jones matrix

This is why we are more interested by the Mueller matrices.

3 Directional statistics : von Mises-Fisher distribution

Let's take an example to understand why circular statistics were introduced.
If we have two angles : 25° and 335°, the average of the two is 180° (linearly).
And if we consider now the two angles : 25° and -25°, these are the same as previously because we have a periodicity of 360°. The average of these two is 0°. But $0° \neq 180°$.

This is why we introduce circular statistics : because of the periodicity.

3.1 Theory

3.1.1 Probability density function

[13] The *von Mises-Fisher* (VMF) distribution is a probability distribution on the $(p-1)$-sphere in $\mathbb{R}^p$, $p \in \mathbb{N}$.
The probability density function of this distribution for the random p-dimensional unit vector x is given by : $f_p(x; \mu, \kappa) = C_p(\kappa)\exp(\kappa\mu^T x)$. We have :

$$\begin{cases} \kappa & \geq & 0\text{, the concentration parameter} \\ \|\mu\| & = & 1\text{, the mean direction} \\ C_p(\kappa) & = & \frac{\kappa^{\frac{p}{2}-1}}{(2\pi)^{\frac{p}{2}} I_{\frac{p}{2}-1}(\kappa)}\text{, the normalization constant} \end{cases}$$

I_α is the the modified Bessel function of the first kind at order α.
$\frac{1}{\kappa}$ is analogous to the variance σ^2 of normal distribution. When $p = 2$ the von Mises-Fisher distribution is qualified as *circular normal distribution.*

Remark 2. When $\kappa = 0$, the distribution is uniform on the unit sphere.
In the remainder of the report, we will therefore only be interested in cases where $\kappa > 0$.

In our context, we will consider only the last three coordinates of the Stokes vectors, i.e., a $\mathbb{R}^3$ vector. Therefore, we have a von Mises-Fisher distribution on the 2-sphere (an ordinary sphere), which corresponds here to the Poincaré sphere.

We have : $\boxed{\begin{aligned} f_3(x;\mu,\kappa) &= C_3(\kappa)\exp(\kappa\mu^T x) \\ &= \tfrac{\kappa}{4\pi\sinh(\kappa)}\exp(\kappa\mu^T x) = \tfrac{\kappa}{2\pi(\exp(\kappa)-\exp(-\kappa))}\exp(\kappa\mu^T x) \end{aligned}}$.

Proof. We want to calculate the normalization constant for $p = 3$:
$C_3(\kappa) = \frac{\kappa^{\frac{1}{2}}}{(2\pi)^{\frac{3}{2}} I_{\frac{1}{2}}(\kappa)}$.

$$\begin{aligned} I_{\frac{1}{2}}(\kappa) &= \sum_{m=0}^{+\infty} \frac{1}{m!\Gamma(m+\frac{3}{2})} \left(\frac{\kappa}{2}\right)^{2m+\frac{1}{2}} \\ &= \frac{\sqrt{2}}{\sqrt{\kappa}} \sum_{m=0}^{+\infty} \frac{1}{m!\Gamma(m+\frac{3}{2})} \left(\frac{\kappa}{2}\right)^{2m+1} \end{aligned}$$

For $z \in \mathbb{C}$, we have $\Gamma(z) = \displaystyle\int_0^{+\infty} t^{z-1}e^{-t}\,\mathrm{d}t$. We put $t = u^2 \Rightarrow dt = 2udu$.

$$\begin{aligned} \Gamma(z) &= 2\int_0^{+\infty} (u^2)^{z-1} u e^{-u^2}\,\mathrm{d}u \\ &= 2\int_0^{+\infty} u^{2z-2} u e^{-u^2}\,\mathrm{d}u \\ &= 2\int_0^{+\infty} u^{2z-1} e^{-u^2}\,\mathrm{d}u \end{aligned}$$

In our case, $z = m + \frac{3}{2}$.

$$\begin{aligned} \Gamma\left(m+\frac{3}{2}\right) &= 2\int_0^{+\infty} u^{2m+2} e^{-u^2}\,\mathrm{d}u \\ &= -\int_0^{+\infty} u^{2m+1}(-2ue^{-u^2})\,\mathrm{d}u \end{aligned}$$

By integration by parts :

$$\begin{aligned} \Gamma\left(m+\frac{3}{2}\right) &= -\left[u^{2m+1}e^{-u^2}\right]_0^{+\infty} + \int_0^{+\infty} (2m+1)u^{2m}e^{-u^2}\,\mathrm{d}u \\ &= (2m+1)\int_0^{+\infty} u^{2m}e^{-u^2}\,\mathrm{d}u \end{aligned}$$

But we have : $\displaystyle\int_0^{+\infty} e^{-u^2}\,\mathrm{d}u = \frac{\sqrt{\pi}}{2}$ (Gaussian integral).

Then by performing m integration by parts, we obtain :

$$\begin{aligned}\Gamma\left(m+\frac{3}{2}\right) &= \frac{(2m+1)(2m-1)...1}{2^m}\frac{\sqrt{\pi}}{2} \\ &= \frac{(2m+1)!}{2^m 2^m m!}\frac{\sqrt{\pi}}{2} \\ &= \frac{(2m+1)!\sqrt{\pi}}{2^{2m+1}m!}\end{aligned}$$

We insert it in $I_{\frac{1}{2}}(\kappa)$:

$$\begin{aligned}I_{\frac{1}{2}}(\kappa) &= \frac{\sqrt{2}}{\sqrt{\kappa}}\sum_{m=0}^{+\infty}\frac{1}{m!\Gamma(m+\frac{3}{2})}\left(\frac{\kappa}{2}\right)^{2m+1} \\ &= \frac{\sqrt{2}}{\sqrt{\kappa}}\sum_{m=0}^{+\infty}\frac{1}{m!\frac{(2m+1)!\sqrt{\pi}}{2^{2m+1}m!}}\left(\frac{\kappa}{2}\right)^{2m+1} \\ &= \frac{\sqrt{2}}{\sqrt{\kappa}\sqrt{\pi}}\sum_{m=0}^{+\infty}\frac{\kappa^{2m+1}}{(2m+1)!} \\ &= \frac{\sqrt{2}\mathrm{sinh}(\kappa)}{\sqrt{\kappa}\sqrt{\pi}}\end{aligned}$$

And so we have :

$$\begin{aligned}C_3(\kappa) &= \frac{\kappa^{\frac{1}{2}}}{(2\pi)^{\frac{3}{2}}I_{\frac{1}{2}}(\kappa)} \\ &= \frac{\kappa^{\frac{1}{2}}}{(2\pi)^{\frac{3}{2}}\frac{\sqrt{2}\mathrm{sinh}(\kappa)}{\sqrt{\kappa}\sqrt{\pi}}} \\ &= \frac{\kappa}{4\pi\mathrm{sinh}(\kappa)}\end{aligned}$$

$\square$

3.1.2 Find μ and κ, and link with the DOP

Suppose that we have a sample X of N vectors x_i which follow the law of VMF. $X := \{x_i \in S^2 \mid x_i \text{ follows } f_3(x;\mu,\kappa), \forall i \in [\![1;N]\!], N \in \mathbb{N}\}$.
For find μ and κ, we use the *maximum likelihood estimation* [9] : we will look for μ and κ which maximizes the likelihood function associated with our density function.
The likelihood function from the VMF law is $L(x_i;\mu,\kappa) = f(x_i;\mu,\kappa)$, and when

we have an observation collection (like our case, our set X) : $L(X;\mu,\kappa) = \prod_{i=1}^{N} f(x_i;\mu,\kappa)$, $x_i \in X$.

$$\begin{aligned} L(X;\mu,\kappa) &= \prod_{i=1}^{N} \frac{\kappa}{4\pi\sinh(\kappa)} \exp(\kappa\mu^T x_i) \\ &= \left(\frac{\kappa}{4\pi\sinh(\kappa)}\right)^N \prod_{i=1}^{N} \exp(\kappa\mu^T x_i) \end{aligned}$$

We will be interested in maximizing the log-likelihood :

$$\mathcal{J}(X;\mu,\kappa) = \ln(L(X;\mu,\kappa)) = N\ln\left(\frac{\kappa}{4\pi\sinh(\kappa)}\right) + \sum_{i=1}^{N} \kappa\mu^T x_i$$

with the constraints : $\mu\mu^T = 1$ and $\kappa > 0$.

We use the KKT theorem to assert that :

$$\begin{cases} \nabla\mathcal{J}(X;\mu,\kappa) + \lambda\nabla h(\mu,\kappa) = 0 \\ h(\mu,\kappa) = 1 - \mu^T\mu = 0 \end{cases}, \lambda \in \mathbb{R}$$

$$\begin{cases} \kappa\sum_{i=1}^{N} x_1^i - 2\lambda\mu_1 = 0 \\ \kappa\sum_{i=1}^{N} x_2^i - 2\lambda\mu_2 = 0 \\ \kappa\sum_{i=1}^{N} x_3^i - 2\lambda\mu_3 = 0 \\ N\left(\frac{1}{\kappa} - \frac{e^{\kappa}+e^{-\kappa}}{e^{\kappa}-e^{-\kappa}}\right) + \sum_{i=1}^{N}\mu^T x_i = 0 \end{cases} \Rightarrow \begin{cases} \mu = \frac{\kappa}{2\lambda}\sum_{i=1}^{N} x_i \\ \mu^T\mu = 1 \\ N\left(\frac{1}{\kappa} - \frac{e^{\kappa}+e^{-\kappa}}{e^{\kappa}-e^{-\kappa}}\right) = -\sum_{i=1}^{N}\mu^T x_i \end{cases}$$

We pose $r = \sum_{i=1}^{N} x_i$.

$$\begin{cases} \mu = \frac{\kappa}{2\lambda} r \\ \mu^T\mu = 1 \\ N\left(\frac{1}{\kappa} - \frac{1}{\tanh(\kappa)}\right) = -\mu^T r \end{cases}$$

Because μ is unitary ($\mu^T\mu = 1$), we have :

1. $\lambda = \frac{\kappa}{2}\|r\|$
2. $N\left(\frac{1}{\kappa} - \frac{1}{\tanh(\kappa)}\right) = -\mu^T r = -\mu^T\mu\|r\| = -\|r\|$
   So κ is the solution of : $\frac{1}{\kappa} - \frac{1}{\tanh(\kappa)} = -\frac{\|r\|}{N}$.

We therefore have the expression of the degree of polarization as a function of κ :

$$\boxed{DOP(\kappa) = \frac{1}{\tanh(\kappa)} - \frac{1}{\kappa}} \tag{1}$$

3.2 Programming

All codes have been done in **Matlab**.
The codes use a code from Florian Pfaff available on Github [8]. He created a class containing several methods to perform statistical calculations for the VMF law. We just modified the code to remove methods that used other files in its directory and which were not useful to us.

3.2.1 `VMFDistributionPoincareSphere.m`

`VMFDistributionPoincareSphere.m` generates vectors which follow a VMF distribution law.
The function has 3 parameters : μ, κ and n the number of vectors that we want to generate by κ. It creates a table containing the generated vectors and plots the representation of these vectors on the Poincaré sphere. The function can take a list of κ and will plot the different vectors on the same Poincaré sphere.

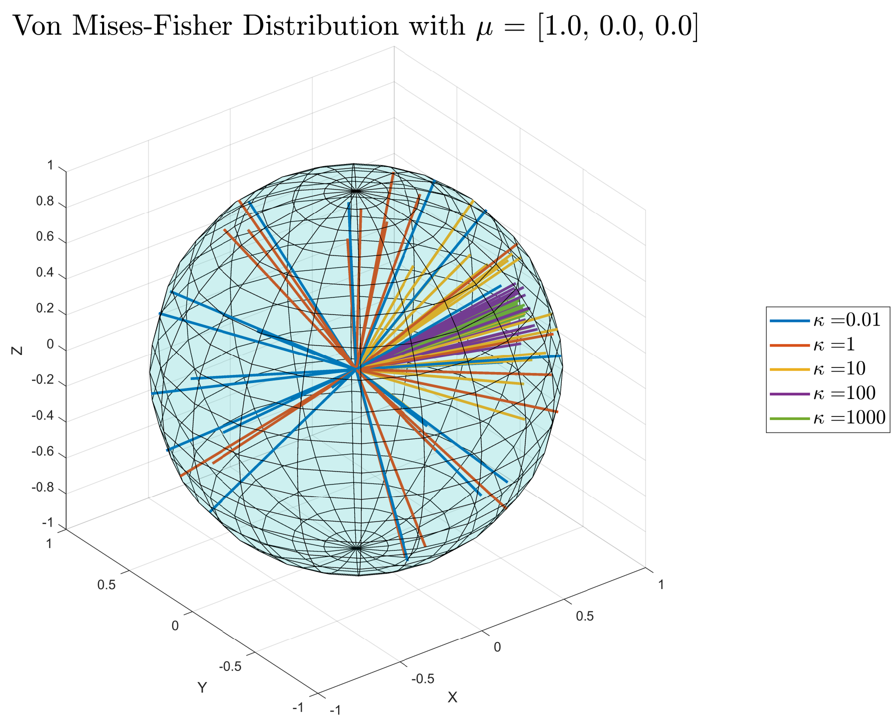

Figure 4: The Poincare Sphere with vectors generate by the VMF distribution for $\mu = (1; 0; 0)$, $\kappa \in \{0.01, 1, 10, 100, 1000\}$ and $n = 20$.

In this representation, we see that more $\kappa \to +\infty$ (the concentration parameter), less the vectors are disperse with respect to μ (the mean direction).

3.2.2 `VMFDegPolazization.m`

We remind that the degree of polarization is defined as being the norm of the Stokes vector.
`VMFDegPolazization.m` plots the degree of polarization (DOP) value as a function of $\frac{1}{\kappa}$. The function has parameters : κ_{min}, κ_{max}, μ the mean direction, n

the number of κ and s the number of vectors on each sample for each κ. For find it, the function create for each κ s vectors which follow the VMF distribution and calculate the norm of the mean vectors.

We have the expression of the DOP as a function of κ 1, we will see if what we found coincide with the data calculated by our algorithms :

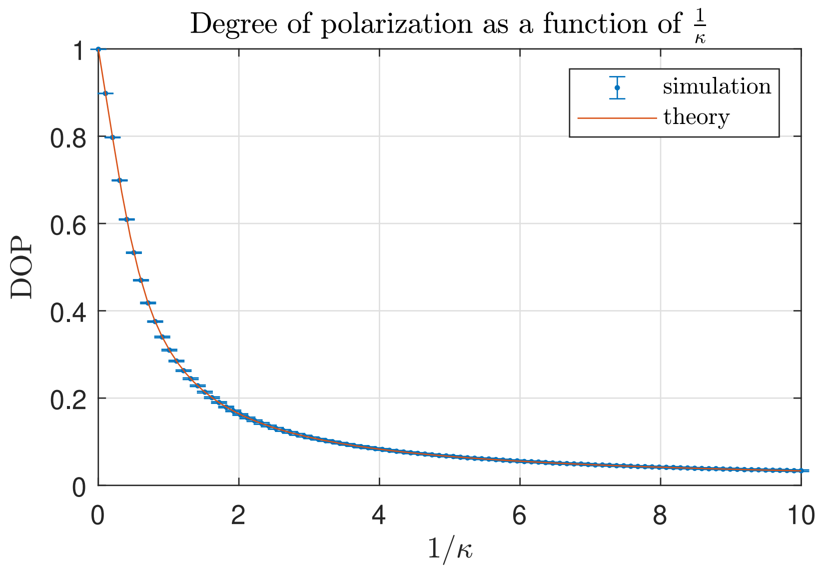

Figure 5: The DOP as a function of $\frac{1}{\kappa}$ for $\mu = (1;0;0)$ and $\kappa \in [0.1;1000]$ and $(n,t,s) = (100{,}100{,}100000)$.

As we can see on the figure 5, the theoretical curve, here : $f_{theo}\left(\frac{1}{\kappa}\right) = \frac{1}{\tanh\left(\frac{1}{\kappa}\right)} - \kappa$, is well between the error limits of the simulations. This allows us to affirm that our formula for DOP as a function of kappa is consistent.

3.2.3 `VMFAngle.m`

We remark that when $\kappa \to +\infty$ (the concentration parameter), more the vectors generate with the VMF distribution are close to μ (the mean direction).
`VMFAngle.m` plots the angle between μ and the mean vector of a sample for each κ. The function has parameters : μ the mean direction, n the number of κ, t the number of test for each κ and s the number of vectors on each sample.

We want to calculate the angle between μ_{theo} given by the user and μ_{cal} the mean vector of the sample.
We have : $\langle \mu_{theo}, \mu_{cal} \rangle = \|\mu_{theo}\| \|\mu_{cal}\| \cos(\widehat{\mu_{theo}, \mu_{cal}})$
$\Rightarrow \cos(\widehat{\mu_{theo}, \mu_{cal}}) = \frac{\langle \mu_{theo}, \mu_{cal} \rangle}{\|\mu_{theo}\| \|\mu_{cal}\|}$

$\Rightarrow \widehat{\mu_{theo}, \mu_{cal}} = \arccos\left(\frac{\langle \mu_{theo}, \mu_{cal}\rangle}{\|\mu_{theo}\| \|\mu_{cal}\|}\right)$

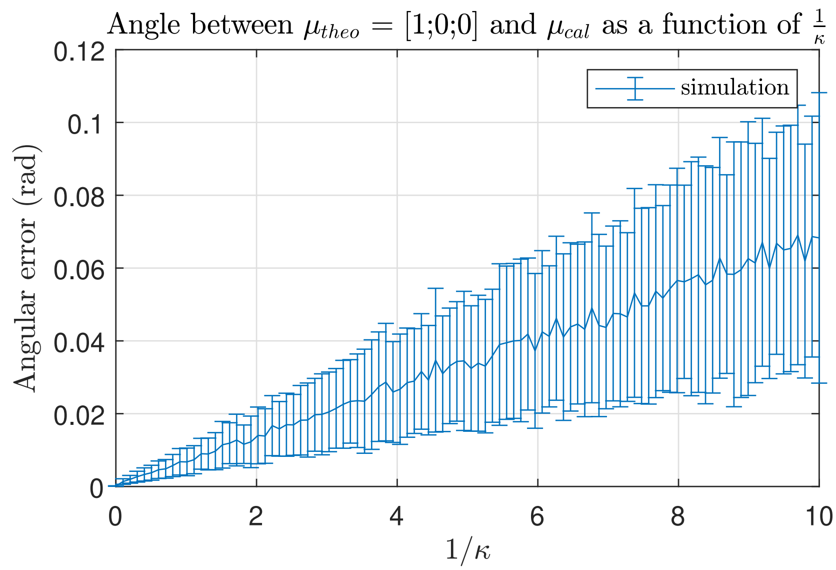

Figure 6: The angular error as a function of $\frac{1}{\kappa}$, for $\mu = (1;0;0)$ and $\kappa \in [0.1;1000]$ and $(n,t,s) = (100,100,100000)$.

4 Fisher-Bingham distribution

We will now focus on another directional distribution law : the Fisher-Bingham (FB) distribution.

[6] The FB distribution is define on the unite spehre S^p, $p \in \mathbb{N}$. Its density function is given by : $f(x;\mu,\Sigma) = C(\mu,\Sigma)^{-1} \exp\left[-(x-\mu)^T \Sigma (x-\mu)\right]$. We have :

$$\begin{cases} x & \text{, an unit vector} \\ \mu & \text{, } \in \mathbb{R}^{p+1} \text{ the mean direction} \\ \Sigma & \text{, the covariance} \\ C(\mu,\Sigma) & \text{, the constant of normalization} \end{cases}$$

This law has two special cases : the Kent distribution (or FB5) distribution and Bingham distribution.

1. The *FB5 distribution* is the distribution on the S^2 in $\mathbb{R}^3$.
2. The *Bingham distribution* is the FB distribution with $\mu = 0$.

We will focus on the *FB5 distribution*.

4.1 Theory

[10] The *Kent* or *FB5 distribution* is the distribution of FB on the S^2 in $\mathbb{R}^3$. We call the Kent distribution FB5 because it's the FB distribution with 5 parameters. The density function of this distribution for the random unit vector $x \in \mathbb{R}^3$ is given by :

$$f_5(x;\kappa,\beta,\gamma_1,\gamma_2,\gamma_3) = \frac{1}{c(\kappa,\beta)} \exp\left[\kappa\gamma_1^T \cdot x + \beta\left[(\gamma_2^T \cdot x)^2 - (\gamma_3^T \cdot x)^2\right]\right].$$

We have :

$$\left\{\begin{array}{cl} c(\kappa,\beta) & \text{, the constant of normalization} \\ \kappa > 0 & \text{, the concentration parameter} \\ 0 \leq 2\beta < \kappa & \text{, determines the ellipticity of the contours of equal probability} \\ \gamma_1 & \text{, the mean direction} \\ \gamma_2,\, \gamma_3 & \text{, the major and minor axes} \end{array}\right.$$

The matrix M given by $M = (\gamma_1, \gamma_2, \gamma_3)$ ($\in M_{3,3}(\mathbb{C})$) must be orthogonal.

Remark 3. If $\beta = 0$, the density function is that on of VMF.

The expression of the constant of normalization c is given by :

$$c(\kappa,\beta) = 2\pi \sum_{j=0}^{\infty} \frac{\Gamma\left(j+\frac{1}{2}\right)}{\Gamma\left(j+1\right)} \beta^{2j} \left(\frac{1}{2}\kappa\right)^{-2j-\frac{1}{2}} I_{2j+\frac{1}{2}}(\kappa).$$

where : Γ is the gamma function and $I_{2j+\frac{1}{2}}$ the modified Bessel function, like in the expression of the normalization constant of VMF distribution.

4.1.1 Reformulation of the density function [4]

A point $\underline{\tilde{x}}$ on the unit sphere is given by : $\underline{\tilde{x}} = (\sin\nu\cos\phi, \sin\nu\sin\phi, \cos\nu)^T$, $\nu \in [0,\pi]$ is the co-latitude and $\phi \in [0, 2\pi]$ is the longitude.
We remind that the matrix M given by $M = (\gamma_1, \gamma_2, \gamma_3)$ ($\in M_{3,3}(\mathbb{C})$) is orthogonal. Our goal is to transforms the original xyz system into the orthogonal system defined by M. To do this, we are going to do 2 rotations.

1. The first is to send the Z axis (north pole $\underline{\tilde{N}}$) on γ_3. This rotation is given by 2 angles : ν_μ and ϕ_μ. We denote by γ_1^1 and γ_2^1, γ_1 and γ_2 which were transformed by the first rotation.

2. The second consists of rotating the plane around γ_3 in order to make X axis coincide with γ_1 rotated and Y axis with γ_2 rotated. This rotation is given by one angle ψ.

Remark 4. We don't have the uniqueness for this transformation.

Remark 5. In the proof, we will use the symbol $\cong$ to signify an equality up to a constant.

We will apply this two rotations on our density function :

$$f_5(\underline{\tilde{x}}; \kappa, \beta, M) \cong \exp\left[\kappa\gamma_3 \cdot \underline{\tilde{x}} + \beta\left((\gamma_1 \cdot \underline{\tilde{x}})^2 - (\gamma_2 \cdot \underline{\tilde{x}})^2\right)\right]$$

The first rotation to send $(0,0,1)^T$ in to γ_3.

$$f_5(\underline{\tilde{x}}; \kappa, \beta, \underline{\tilde{N}}, \psi) \cong \exp\left[\kappa\tilde{x_3} + \beta\left((\gamma_1^1 \cdot \underline{\tilde{x}})^2 - (\gamma_2^1 \cdot \underline{\tilde{x}})^2\right)\right]$$

The second rotation to make γ_1^1 coincide with X axis and γ_2^1 with Y axis. The angle of this rotation is denote by ψ in the code.

$$f_5(\underline{\tilde{x}}; \kappa, \beta, \underline{\tilde{N}}, 0) \cong \exp\left[\kappa\tilde{x_3} + \beta\left(\tilde{x_1}^2 - \tilde{x_2}^2\right)\right]$$

This formula is call *canonical form*, and we have the notation $f_5(\underline{\tilde{x}}; \kappa, \beta, \underline{\tilde{N}}, 0) = f_5(\underline{\tilde{x}}; \kappa, \beta)$.
The expression of $\underline{\tilde{x}}$ is given by $\underline{\tilde{x}} = (\sin\nu\cos\phi, \sin\nu\sin\phi, \cos\nu)^T$.

$$\begin{aligned} f_5(\underline{\tilde{x}}; \kappa, \beta) &\cong \exp\left[\kappa\cos\nu + \beta\left((\sin\nu\cos\phi)^2 - (\sin\nu\sin\phi)^2\right)\right] \\ &\cong \exp\left[\kappa\cos\nu + \beta\sin^2\nu\cos 2\phi\right] \end{aligned}$$

We pose $x = \cos\nu$:

$$\begin{aligned} g_5(x, \phi; \kappa, \beta) &\cong \exp\left[\kappa x + \beta(1-x^2)\cos 2\phi\right] \\ &\cong \exp(\kappa x)\exp\left[\beta(1-x^2)\cos 2\phi\right] \end{aligned}$$

4.1.2 Link with the DOP

We were unable to establish a link between FB5 and DOP, the research did not reveal a direct link between the two.

4.2 Programming

All the codes use for the visualisation of the FB5 distribution are from G.Terdik, B.Wainwright and S.R. Jammalamadaka [3].

4.2.1 `FB5DistributionSphere.m`

`FB5DistributionSphere.m` generate points on Poincaré sphere which follow a FB5 distribution law.
The function has 5 parameters ; κ, β, μ, ψ and n. κ and β are the parameters of the distribution law, the function can take a list of κ and β. μ is the mean direction. ψ is the angle of rotation to make the axes coincide with X and Y. And n is the number of vectors that we want to generate by couples of (κ, β).

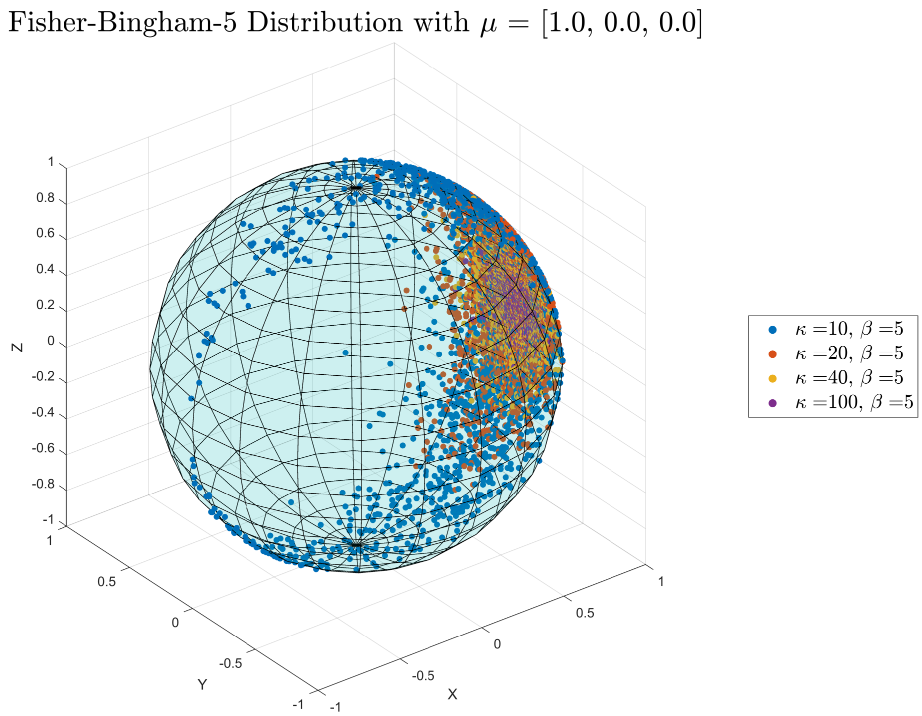

Figure 7: The Poincare Sphere with points generate by the FB5 distribution for $\mu = (1; 0; 0)$, $\kappa \in \{10, 20, 40, 100\}$, $\beta = 5$, $\psi = 0$ and $n = 2000$.

For fixed β, we see that if $\kappa \to +\infty$, the points concentrated around the mean direction. And when $\kappa \to 2\beta$, the points form an ellipse whose center is the mean direction. The value of κ influences the dispersion of points. The parameter ψ rotates the points generated by the distribution function around the mean direction.

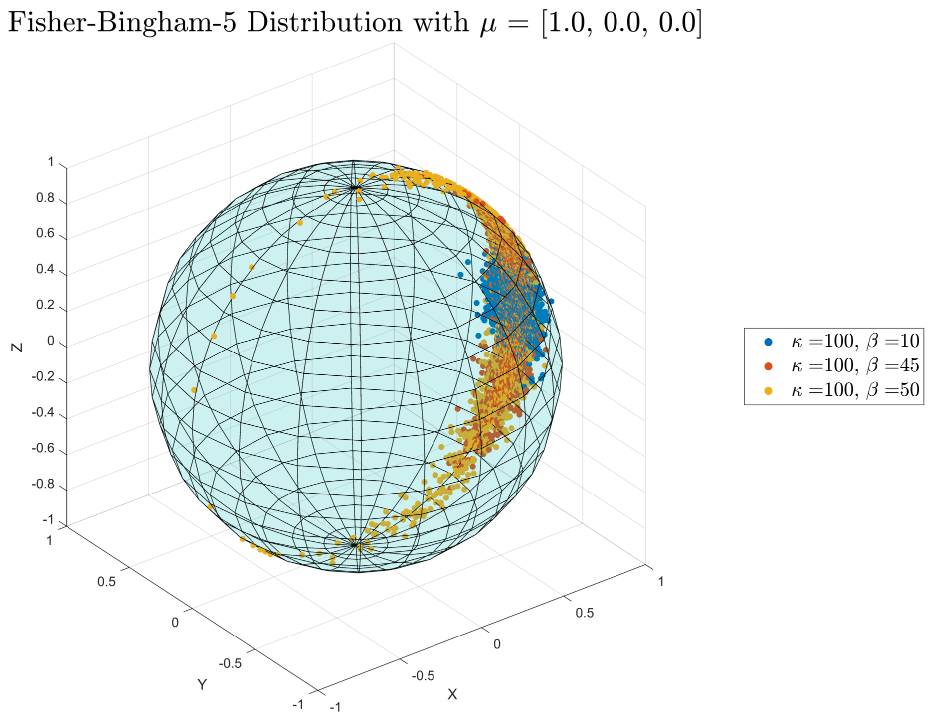

Figure 8: The Poincare Sphere with points generate by the FB5 distribution for $\mu = (1; 0; 0)$, $\kappa = 100$, $\beta \in \{10, 45, 50\}$, $\psi = 0$ and $n = 2000$.

For fixed κ, we see that if $\beta \to 0$, the points form a disk around the mean direction like in the VMF distribution law. And when $\beta \to \frac{1}{2}\kappa$, the points form an ellipse whose center is the mean direction. The value of β influences the dispersion of points relative to the ellipse.

4.2.2 `FB5DegPolarization.m`

`FB5DegPolarization.m` plots the degree of polarization (DOP) values as a function of $\frac{1}{\kappa}$ and $\frac{1}{\beta}$.
The function has 9 parameters : κ_{min}, κ_{max}, β_{min}, β_{max}, μ the mean direction, ψ is the angle of rotation to make the axes coincide with X and Y, n the number of values between κ_{min} and κ_{max} (respectively β_{min} and β_{max}), s the number of vectors on each samples for each (κ, β) and t the number of test for each (κ, β).

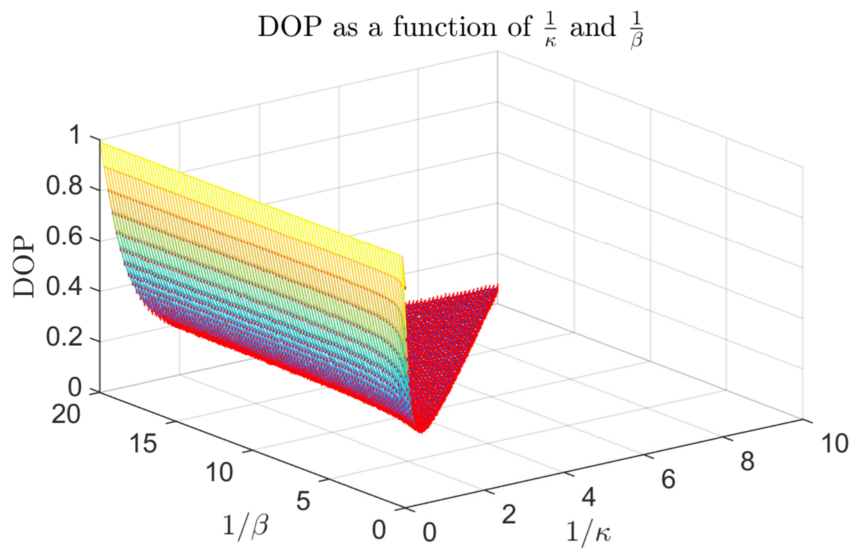

Figure 9: The DOP as a function of $\frac{1}{\kappa}$ and $\frac{1}{\beta}$, for $\mu = (1; 0; 0)$ and $\kappa \in [0.1; 100]$, $\beta \in [0.05; 50]$ and $(n, t, s) = (100,100,1000)$.

In our analysis, we attempted to correlate the results with the DOP equation for VMF 1 under the condition $0 \leq 2\beta < \kappa$. However, this approach did not produce conclusive results.

4.2.3 `FB5Angle.m`

As `VMFAngle.m`, `FB5Angle.m` plots the angle between the mean direction μ and the mean vector of a sample for each couples (κ, β). The function has 5 parameters : μ the mean direction, ψ is the angle of rotation to make the axes coincide with X and Y, n the number of values between κ_{min} and κ_{max} (respectively β_{min} and β_{max}), s the number of vectors on each samples for each (κ, β) and t the number of test for each (κ, β).

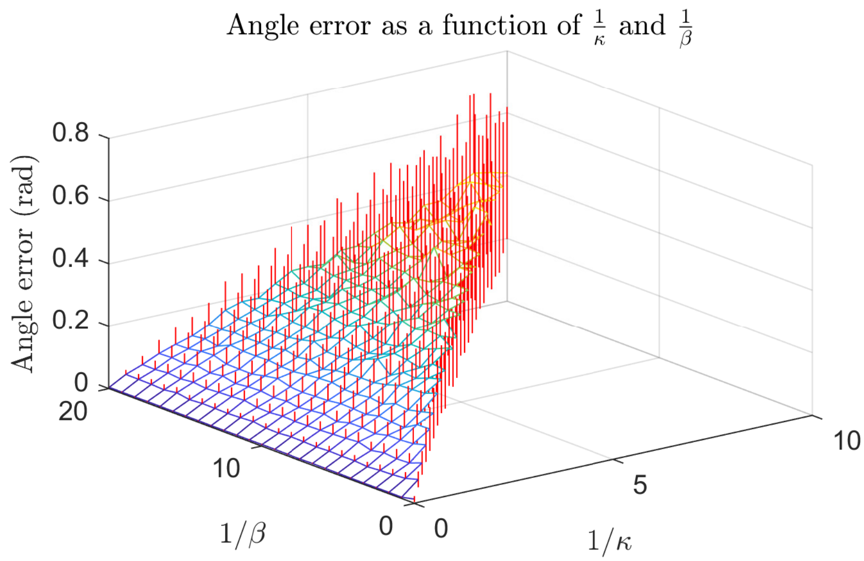

Figure 10: The Angle error as a function of $\frac{1}{\kappa}$ and $\frac{1}{\beta}$, for $\mu = (1; 0; 0)$ and $\kappa \in [0.1; 100]$, $\beta \in [0.05; 50]$ and $(n, t, s) = (25{,}100{,}1000)$.

5 Clustering algorithm

The most well-known clustering algorithm is k-means. Following a sample X of data and a cluster number k, the algorithm associates to each data a cluster. It's consider as a hard clustering.
Fuzzy C-means (*fcm*) is also a clustering algorithm. It's consider as a soft clustering, the data point can belong to more than one cluster : the probability that each data point belongs to a cluster is given.

5.1 Comparison of k-means and fcm

We have a program, `image_segmentation.m`, that performs a segmentation once with the k-means algorithm and another time with the fcm algorithm on the same image.
By running it we realized two things : the segmentation for k-means and fcm differs, and if you run the program with the same parameter you can get a new segmentation for each methods.

5.1.1 k-means algorithm

[12] Given an initial set of k means $m_1^{(1)}, ..., m_k^{(1)}$.

1. **Assignment step :** Assign each observation to the cluster with the nearest mean : that with the least squared Euclidean distance.

$S_i^{(t)} = \{x_p : \|x_p - m_i^{(t)}\|^2 \le \|x_p - m_j^{(t)}\|^2 \ \forall j, j \in [\![1;k]\!]\}$.

2. **Update step :** Recalculate means for observations assigned to each cluster.
   $m_i^{(t)} = \frac{1}{|S_i^{(t)}|} \sum_{x_j \in S_i^{(t)}} x_j$.

5.1.2 Fuzzy C-means algorithm

[2] Let $X = \{x_1, ..., x_n\}$ a finite collection of n elements and $C = \{c_1, ..., c_c\}$ a list of c cluster centers.

1. Choose random centroid at least 2 and put values to them randomly.
2. Compute membership matrix : $U_{ij} = \frac{1}{\sum_{k=1}^{c} \left(\frac{|x_i - c_j|}{x_i - c_k}\right)^{\frac{2}{m-1}}}$, $m > 1$ the fuzziness parameter (generally taken as 2).
3. Calculate the clusters centers : $C = \frac{\sum_{i=1}^{n} U_{ij}^m . x_i}{\sum_{i=1}^{n} U_{ij}^m}$.
4. If $C^{(k-1)} - C^{(k)} < \epsilon$ then STOP, else go to 2.

6 Mueller imager

We have : $I_{in} = A.S_{in}$, with : I_{in} our input intensity vector, A such that A has for coordinates the vertices of a regular tetrahedron inscribed in a sphere and S_{in} our input Stokes vector.
In each pixel, we have : $I_{in,meas} = I_{in} + \epsilon$ with ϵ the noise (the error that the different measurements can provide). And that we have : $S_{in,meas} = A^{-1} I_{meas}$.

6.1 `Stokes_segmentation.m`

We suppose that we have three ares to segment and that S_{z_0}, S_{z_1} and S_{z_2} are the Stokes vectors of these zones. The program function has eight parameters : the Stokes vectors of our zones, the theoretical input Stokes image, the index of the zones, a matrix A as at the beginning of the paragraph, the variance of Gaussian noise and the method (fcm or k-means).

- If we don't have theoretical input Stokes image then the program proposes to draw a polygon and an ellipse to create the theoretical input Stokes image.
- A can be calculated from the program `PolyhedreRegulier.m`.

The program performs segmentation based on Stokes measurements of a test image using two clustering methods : fcm and k-means. It adds a Gaussian noise to the input image data. Then, by the chosen clustering method (fcm or k-means), it segments the data and calculates the centers of the clusters. The results are displayed on segmented images and Poincaré spheres, allowing an analysis of the distributions and clusters of the segmented points.

6.2 Analysis of results

For the analysis of the results we will take for all cases a variance of 0.01 and choose the method of segmentation fcm.

6.2.1 Stokes vector close

We have the coordinates of three stokes vectors that are close : $(S_{z_0}, S_{z_1}, S_{z_2}) = ((1,1,0,0)^T, (1, 0.995, -0.0998, 0)^T, (1, 0.995, 0.0998, 0)^T)$ and Gaussian noise variance is 0.01.

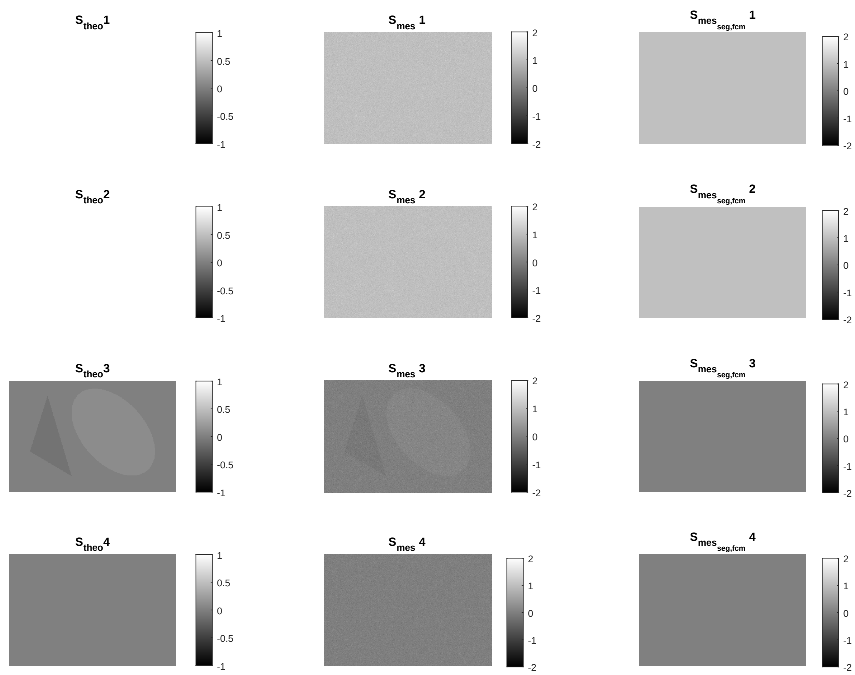

Figure 11: The coordinate of each pixel theoretical, measured (with noise) and after application of the fcm segmentation for vectors $(S_{z_0}, S_{z_1}, S_{z_2}) = ((1,1,0,0)^T, (1, 0.995, -0.0998, 0)^T, (1, 0.995, 0.0998, 0)^T)$ and a variance of 0.01.

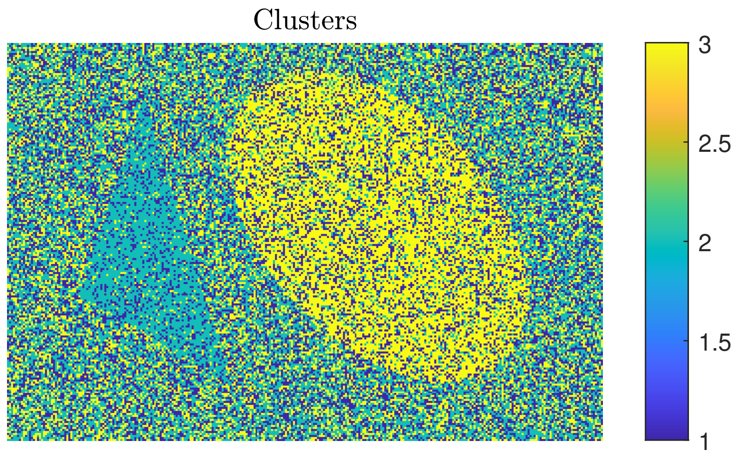

Figure 12: The clusters selects for each pixel by the method fcm for vectors $(S_{z_0}, S_{z_1}, S_{z_2}) = ((1, 1, 0, 0)^T, (1, 0.995, -0.0998, 0)^T, (1, 0.995, 0.0998, 0)^T)$ and a variance of 0.01.

As we can see, although noise is not a major variance for close Stokes vectors the segmentation has trouble assigning clusters. We can see the demarcation of the zones with our naked eyes but the zones are still very troubled.

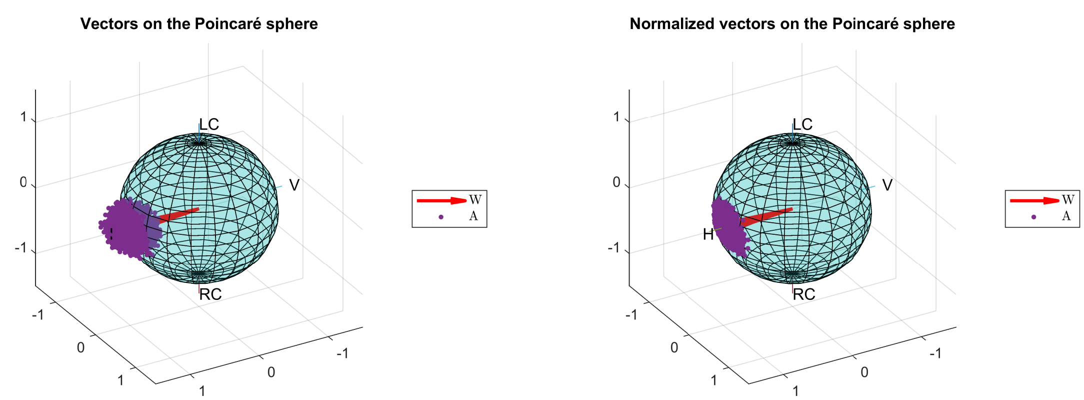

Figure 13: The clusters selects for each pixel by the method fcm for vectors $(S_{z_0}, S_{z_1}, S_{z_2}) = ((1, 1, 0, 0)^T, (1, 0.995, -0.0998, 0)^T, (1, 0.995, 0.0998, 0)^T)$ and a variance of 0.01.

The vectors obtained by segmentation are placed on the Poincaré sphere, they form balls around the initial Stokes vector. We normalized them and posted on the sphere again.

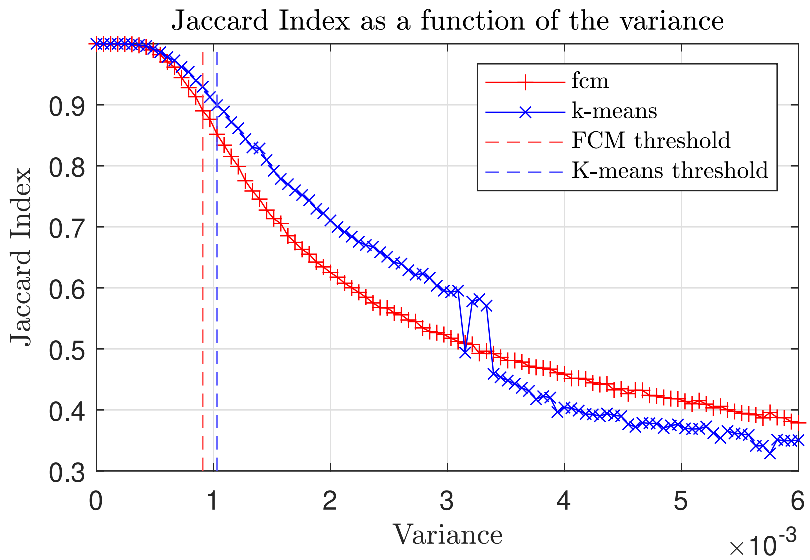

Figure 14: The Jaccard Index as a function of the variance for vectors $(S_{z_0}, S_{z_1}, S_{z_2}) = ((1,1,0,0)^T, (1,0.995,-0.0998,0)^T, (1,0.995,0.0998,0)^T)$ with thresholds for which the Jaccard index is less than 0.9.

As can be seen on the curve to obtain a Jaccard index greater than 0.9 for this case it would have to have a variance of less than 9.10^{-4} for the fcm segmentation. We have disturbances for the kmeans curve because the algorithm stops matlab of the k-means stops after 100 iterations even if there was no convergence.

6.2.2 Orthogonal Stokes vectors

We have : $(S_{z_0}, S_{z_1}, S_{z_2}) = ((1,1,0,0)^T, (1,0,1,0)^T, (1,0,0,1)^T)$: orthogonal Stokes vectors and Gaussian noise variance is 0.01.

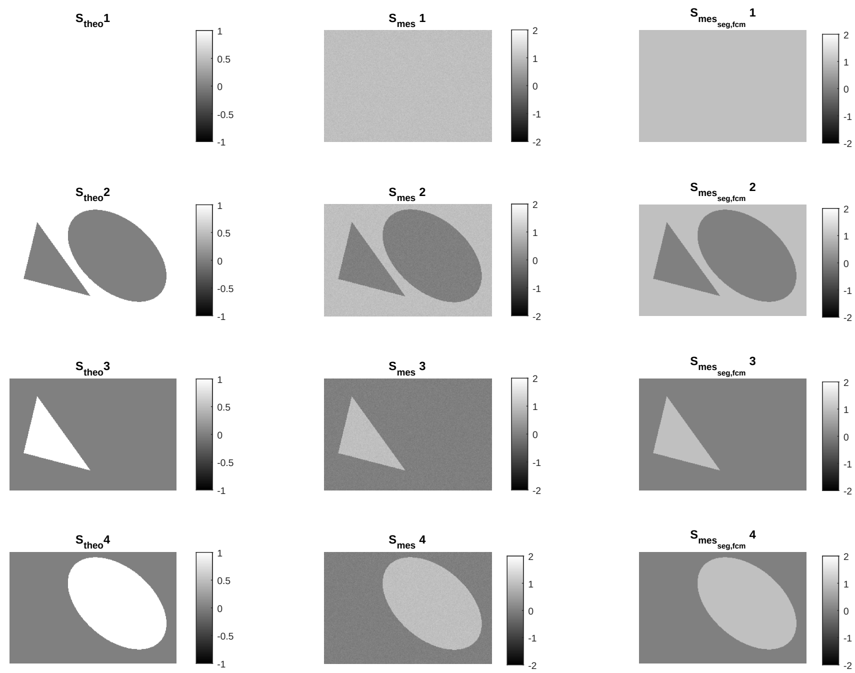

Figure 15: The coordinate of each pixel theoretical, measured (with noise) and after application of the fcm segmentation for vectors $(S_{z_0}, S_{z_1}, S_{z_2}) = ((1,1,0,0)^T, (1,0,1,0)^T, (1,0,0,1)^T)$ and a variance of 0.01.

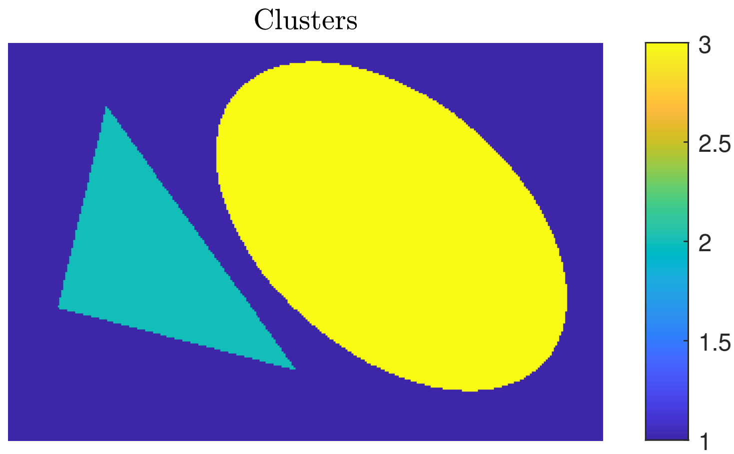

Figure 16: The clusters selects for each pixel by the method fcm for vectors $(S_{z_0}, S_{z_1}, S_{z_2}) = ((1,1,0,0)^T, (1,0,1,0)^T, (1,0,0,1)^T)$ and a variance of 0.01.

It can already be seen by eye that segmentation gives results closer to the theoretical compared to the previous case where vectors were close.

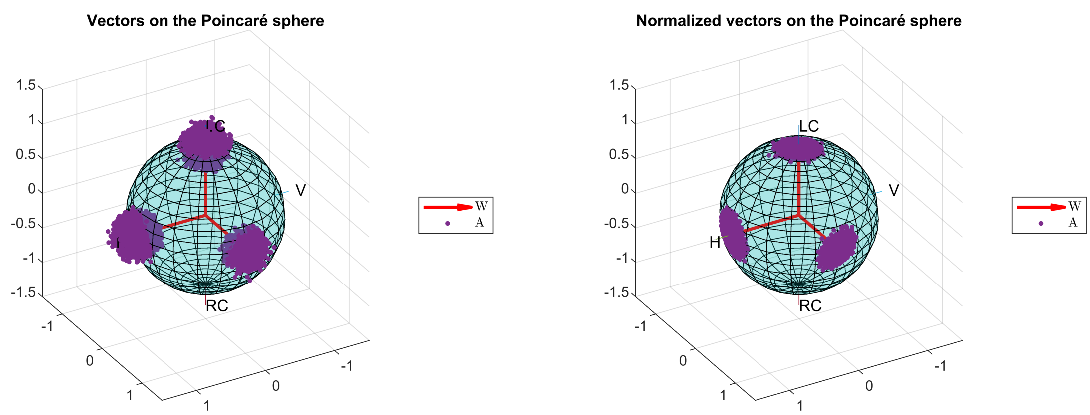

Figure 17: The clusters selects for each pixel by the method fcm for vectors $(S_{z_0}, S_{z_1}, S_{z_2}) = ((1,1,0,0)^T, (1,0,1,0)^T, (1,0,0,1)^T)$ and a variance of 0.01.

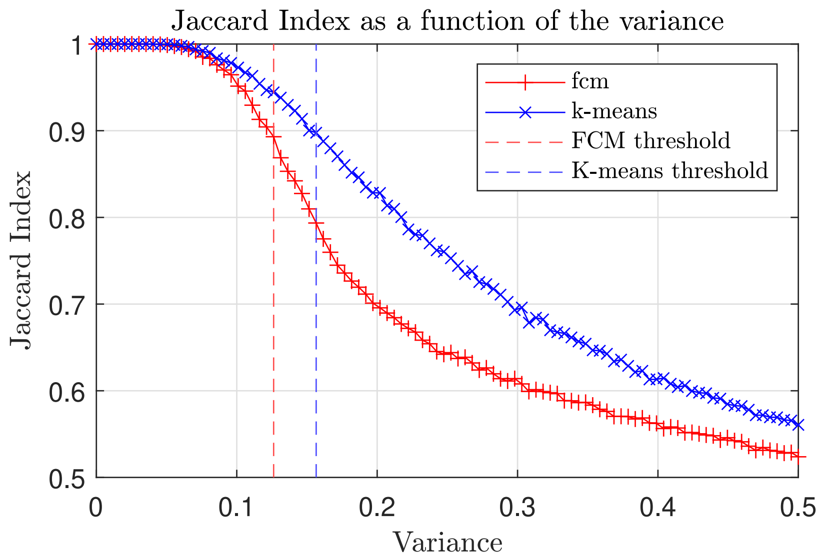

Figure 18: The Jaccard Index as a function of the variance for vectors $(S_{z_0}, S_{z_1}, S_{z_2}) = ((1,1,0,0)^T, (1,0,1,0)^T, (1,0,0,1)^T)$ with thresholds for which the Jaccard index is less than 0.9.

The value of the variance for which the Jaccard index is below 0.9 is 0.13, this is higher than for the case where the Stokes vectors were close. It can therefore be assumed that the stability of the segmentation depends on the angle between the Stokes vectors.

Link with VMF

To make the connection with the statistical distribution of VMF we will look at the points generated around the vector $(1,0,0,1)^T$ for the orthogonal Stokes vectors, i.e. the points which have been classified in the area following this Stokes vector.

We remind that we have : $\boxed{\begin{aligned} f_3(X;\mu,\kappa) &= C_3(\kappa)\exp(\kappa\mu^T X) \\ &= \tfrac{\kappa}{4\pi\sinh(\kappa)}\exp(\kappa\mu^T X) \end{aligned}}$.

As the vectors have been normalized, we have also : $z^2 + x^2 + y^2 = 1$, i.e. $z = \sqrt{1 - x^2 - y^2}$. (x, y, z) are the last three coordinates of the Stokes vector.

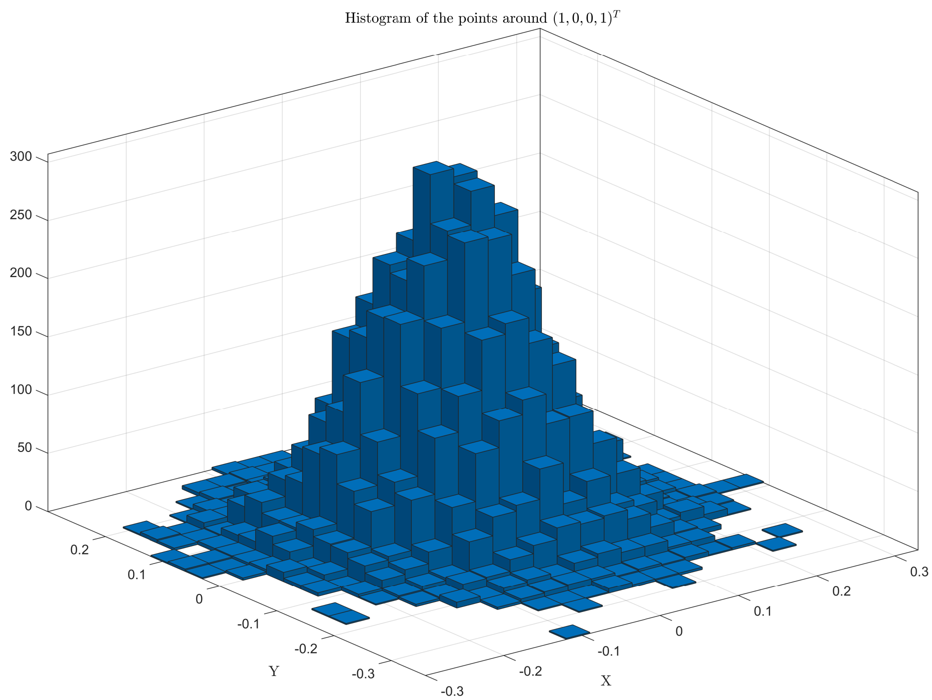

Figure 19: The histogram of the points around $(1,0,0,1)^T$ for vectors $(S_{z_0}, S_{z_1}, S_{z_2}) = ((1,1,0,0)^T, (1,0,1,0)^T, (1,0,0,1)^T)$ and a variance of 0.01.

As we look at the points generated around the middle direction $(1,0,0,1)^T$, we have : $\boxed{\begin{aligned} f_3(X;\mu,\kappa) &= \frac{\kappa}{4\pi\sinh(\kappa)}\exp(\kappa\mu^T X) \\ &= \frac{\kappa}{4\pi\sinh(\kappa)}\exp(\kappa\sqrt{1-x^2-y^2}) \end{aligned}}$.

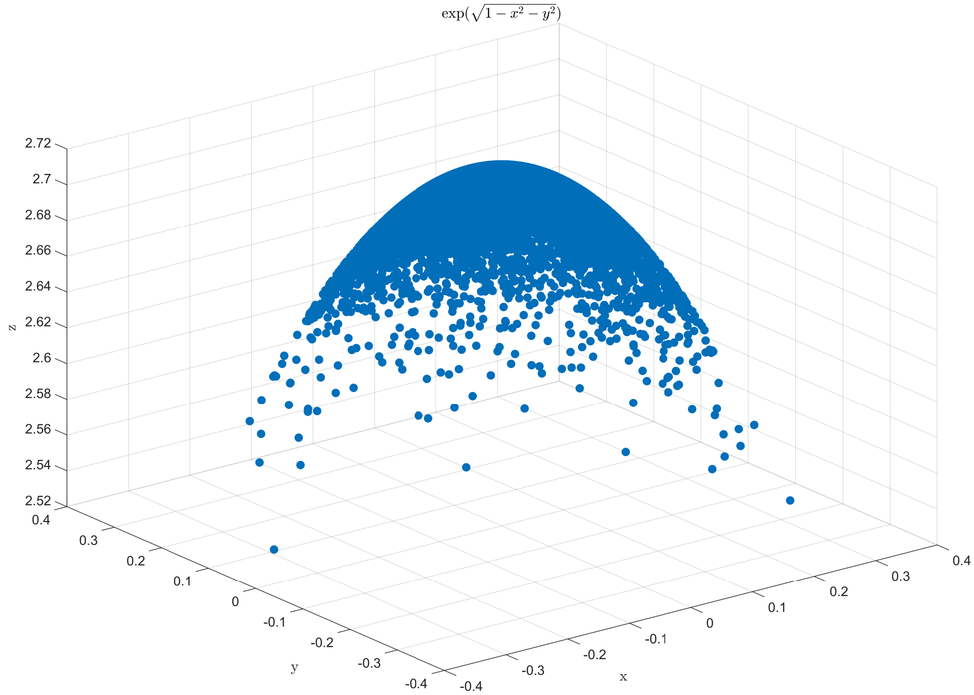

Figure 20: The dot cloud of $\exp(\sqrt{1-x^2-y^2})$ for vectors $(S_{z_0}, S_{z_1}, S_{z_2}) = ((1,1,0,0)^T, (1,0,1,0)^T, (1,0,0,1)^T)$ and a variance of 0.01.

The program `Stokes_segmentation.m` includes `[N, Xedges, Yedges] = histcounts2(points_S_z2(:,1), points_S_z2(:,2));`. `N` is a matrix where each element represents the number of points (frequency) in a bin and `Xedges, Yedges` are the vectors containing the edges of bins (bins) on X and Y axes.

On Matlab by doing:

1. `[XX,YY] = meshgrid((Yedges(1:end-1)+Yedges(2:end))/2,(Xedges(1:end-1)+Xedges(2:end))/2);`
   Calculating the centers of each bin consecutive for x and y axis.

2. `M = N/sum(N(:));`
   Transforms frequency probabilities by normalizing frequencies by the total number of frequencies.

3. Using curve fitting, for $(x, y, z) = (XX, YY, M)$ and an expression model curve : $\frac{0.02*0.02*k}{4*\pi*\sinh(k)} \exp(k * \sqrt{1-x^2-y^2})$. 0.02 is the step in x and y.

We found a R-square value of 0.97 and $k = 160$.

7 Conclusion

During this internship, after acquiring knowledge of polarization base notations, we derived the formula for calculating the Degree of Polarization (DOP) for vectors following a Von Mises-Fisher (VMF) statistical distribution. We also examined another distribution, FB5, but were unable to establish a connection with the DOP. Subsequently, using the FCM segmentation method, we generated images with three distinct zones, each corresponding to different Stokes vectors, and added noise to these images. We then attempted to recover the original image using segmentation techniques. We observed that some points did not lie on the sphere, necessitating normalization. Finally, using curve fitting techniques, we found that with a correlation coefficient greater than 0.97, these points followed a VMF distribution.

The next step involves analyzing results obtained from experimentally acquired polarized images, rather than those generated through Matlab simulations. It is crucial to clearly relate the variance and the angle between Stokes vectors to establish the conditions necessary for achieving acceptable segmentation results. Theoretically, points outside the Poincaré sphere can be observed due to various factors, including measurement noise, errors in calibration, or inaccuracies in the experimental setup. Physically, addressing these outlier points allows for more robust data interpretation and improved accuracy in the segmentation process. By better handling these discrepancies, we can enhance the reliability and usability of the segmentation results derived from the polarized images.

8 Annexe

8.1 Complex conjugate

$a \in \mathbb{C}$, a^* is the complex conjugate of the complex a.
$(x, y) \in \mathbb{R}^2$, $a = x + ib \Rightarrow a^* = x - ib$. $\phi \in \mathbb{R}$, $a = \exp(i\phi) \Rightarrow a^* = \exp(-i\phi)$.

8.2 Transpose

$(n, p) \in \mathbb{N}^2$, $A \in M_{n,p}(\mathbb{C})$. We denote by A^T the transpose matrix of the matrix A.
For $(i, j) \in \mathbb{N}^2$, if $A = (a_{ij})$, we have $A^T = (a_{ji})$.

8.3 Integration by parts

Let $u, v : [a, b] \to \mathbb{C}$ such that $u, v \in \mathcal{C}^1$.
So : $\displaystyle\int_a^b u'(t)v(t)\,\mathrm{d}t = [u(t)v(t)]_a^b - \int_a^b u(t)v'(t)\,\mathrm{d}t$

8.4 KKT Theorem

Let $V \subset \mathbb{R}$ and K the set of the constraints. Let $\mathcal{J}$, h_1, ..., h_p, g_1, ..., g_q $\in \mathcal{C}^1(V)$.
If $\mathcal{J}$ admits at $\bar{x}$ a local minimum, and that the constraints are qualified at this point, so $\exists \lambda_1$, ..., $\lambda_p \in \mathbb{R}$ and η_1, ..., $\eta_q \in \mathbb{R}_+$ not all zero, such that :

$$\nabla \mathcal{J}(\bar{x}) + \sum_{i=1}^{p} \lambda_i \nabla h_i(\bar{x}) + \sum_{j=1}^{q} \eta_j \nabla g_j(\bar{x}) = 0$$
$$\forall j \in [\![1; q]\!],\ \mu_j g_j(\bar{x}) = 0$$

8.5 Dot product

Let $(u, v) \in \mathbb{R}^n$, $n \in \mathbb{N}$, $u = (u_1, u_2, ..., u_n)$ and $v = (v_1, v_2, ..., v_n)$. We denote by $\langle ., . \rangle$ or $. \cdot .$ the dot product between two vectors.
$\langle u, v \rangle = u \cdot v = \|u\|\|v\| \cos(\widehat{u, v}) = \sum_{i=1}^{n} u_i v_i$.

8.6 Orthogonal

$n \in \mathbb{N}$, a matrix $A \in M_{n,n}(\mathbb{R})$ is orthogonal, if we have $A^{-1} = A^T$.

8.7 Jaccard Index

The Jaccard Index is a statistical measure used to quantify the similarity between two sets. It is calculated as the size of the intersection divided by the size of the union of the two sets, giving a value between 0 and 1. A Jaccard Index of 1 indicates that the sets are identical, while a value of 0 means they have no elements in common. [11]

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