Graph Convolutional Networks and some applications

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Introduction

continuation of the previous project

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- discontinuities detection and Burgers' equation
- interpolation problem and linear transport equation

Definitions

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- computes d' dimensional representations of the nodes
- uses recursive neighborhood diffusion and message passing
- each graph node gathers features from its neighbors

Definitions: pooling layers

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inputs: mesh Ω, nodes features X, integer k,
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the new mesh has the k nodes with the highest score

Top-k pooling example

X_i = node position, $p = (1, 1)$

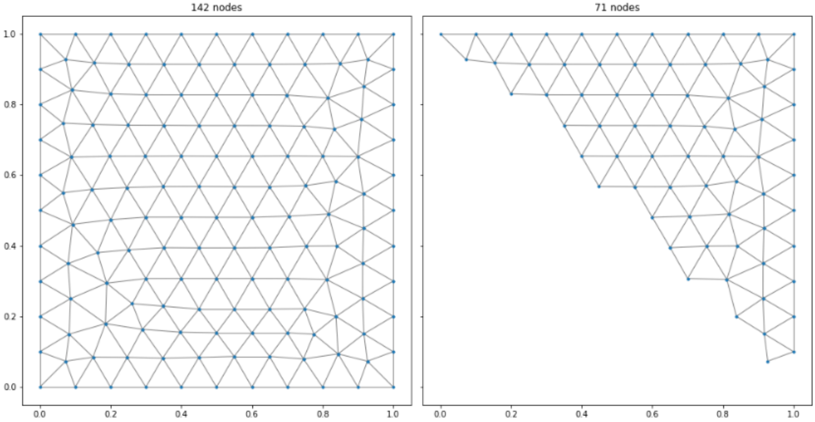

Figure: Initial mesh (left) and pooled mesh (right).

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k-Means example

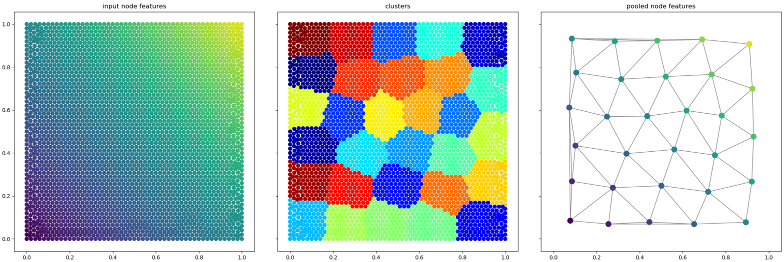

Figure: Initial mesh (left), clusters (center), and pooled mesh (right).

Frontier detection problem

Frontier detection: dataset

problem: detect the frontier between to areas on a mesh

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3 types of areas:

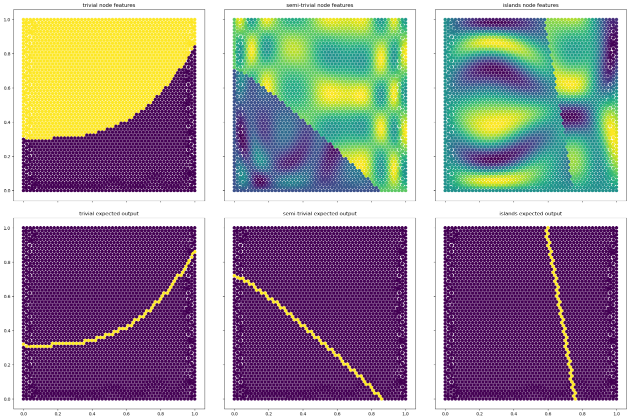

Figure: Trivial dataset (left), semi-trivial dataset (center) and islands dataset (right).

Frontier detection: previous results

simple sequential model: GCN layers put one after the other

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worked only on the trivial dataset

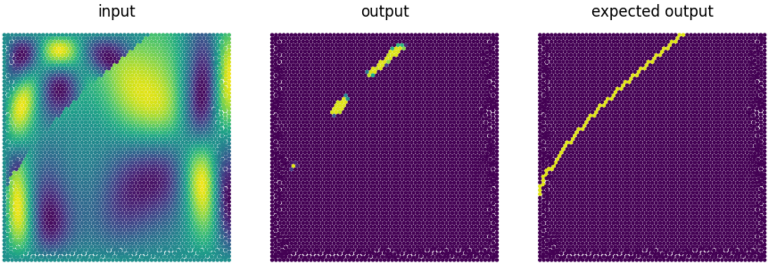

Figure: Old model results on the islands dataset.

Frontier detection: U-Net architecture

more complex architecture

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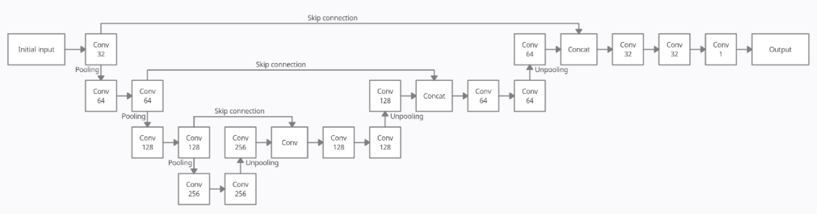

Figure: Architecture of the model used.

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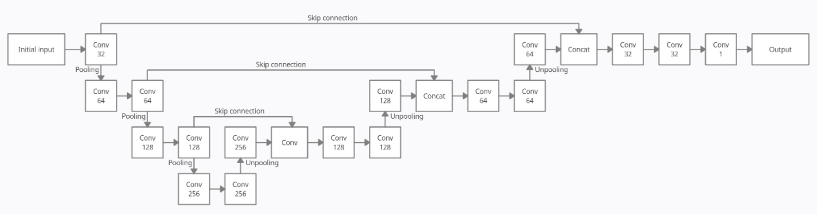

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3 pooling layers and 3 unpooling layers

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first model: U-Net with Vanilla GCN layers and Top-k pooling layers

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⇒ good results on trivial/semi-trivial dataset

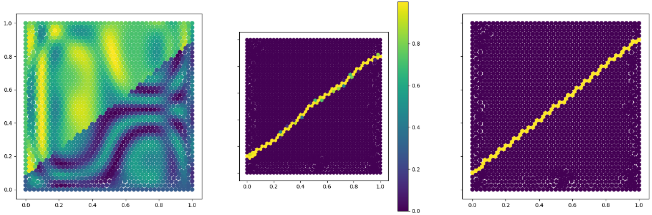

Figure: Input (left), model prediction (middle), expected output (right).

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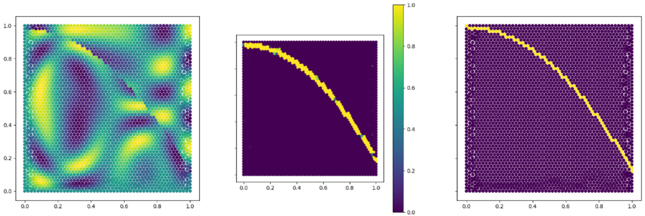

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with $\rho : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$, $a \in \mathbb{R}^2$ and $t \in [0, T]$.

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transforms PDE into algebraic equations

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mesh Ω, triangles Ω_j, t_n discretization of $[0, T]$

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final scheme:

$$\rho_j^{n+1} = \rho_j^n - \frac{\Delta t}{|\Omega_j|} \sum_{k \in E_j} d_{jk} F(\rho_j^n,\ \rho_k^n).$$

where:

$$F(\rho_j^n,\ \rho_k^n) = \frac{1}{2}\Big[a \cdot n_{jk}\left(\rho_j^{n2} + \rho_j^{n2}\right) + \max\left(|a \cdot n_{jk}\rho_j^n|,\ |a \cdot n_{jk}\rho_k^n|\right)(\rho_j^n - \rho_k^n)\Big]$$

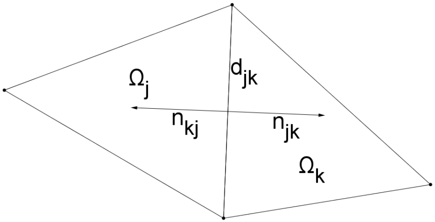

Figure: Notations.

Example of solutions

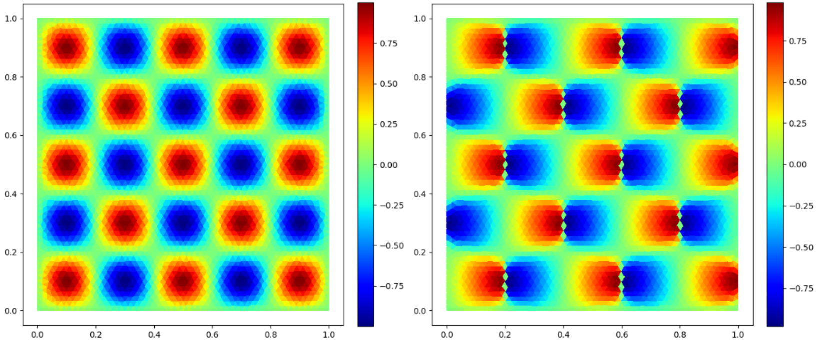

Figure: Initial solution (left) and final solution (right) at $t = 0.05s$, $a = (1, 0)$.

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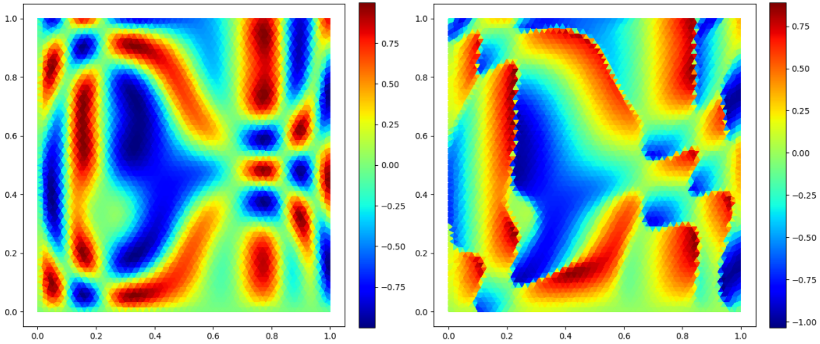

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Dynamic refining

refine the mesh while we are computing the final solution

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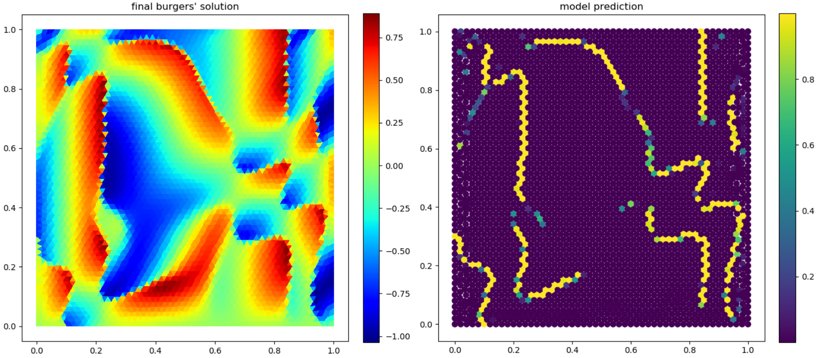

Figure: Final Burgers' solutions (left), and model predictions (right).

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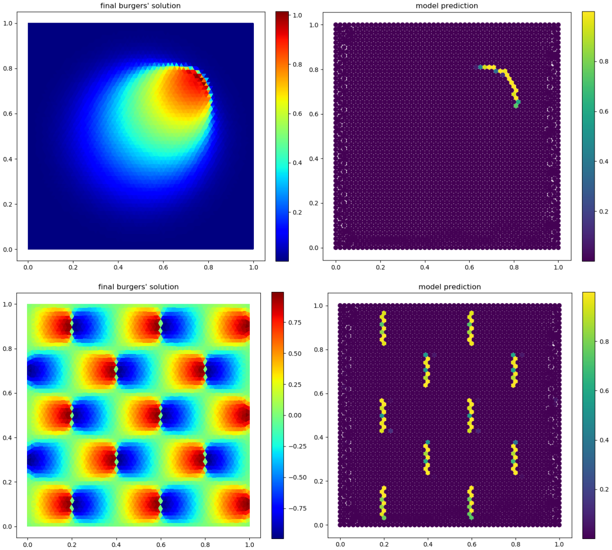

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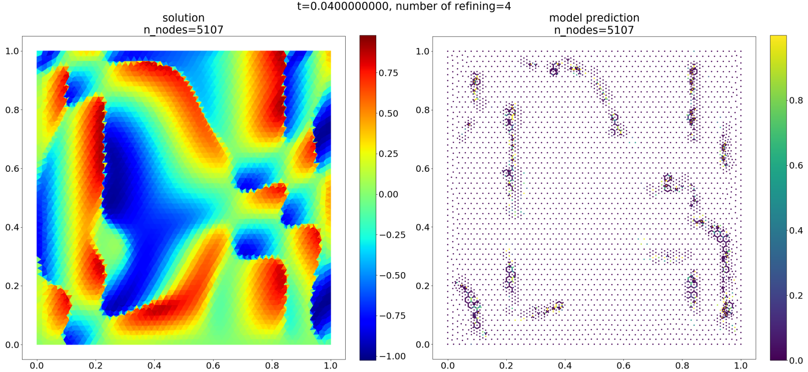

Figure: Solution with refinements (left), and model prediction (right).

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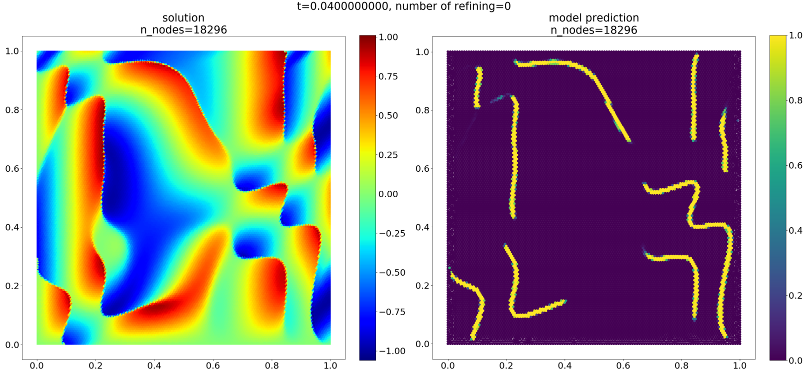

Figure: Finer solution (left), and model prediction (right).

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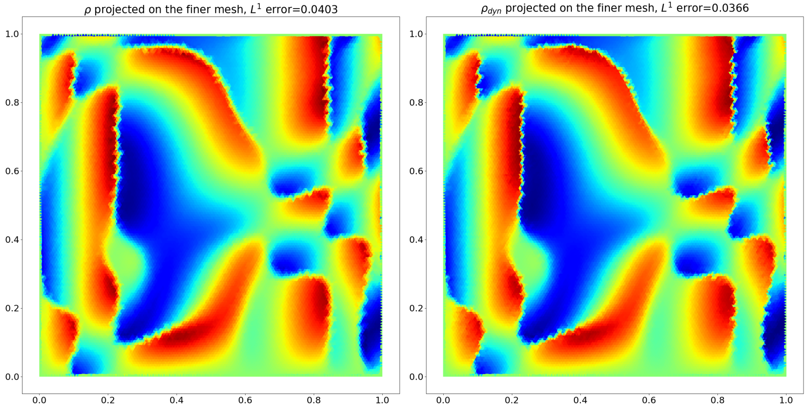

Figure: Projections and errors on the finer mesh.

Transport equation and interpolation problem

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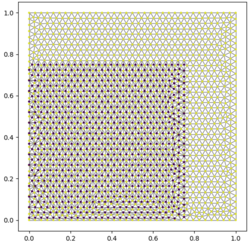

Interpolation problem

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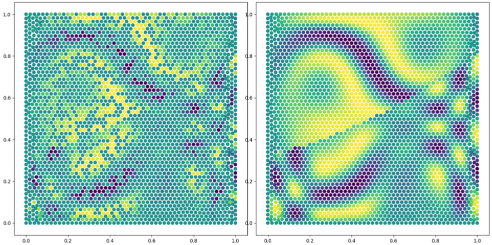

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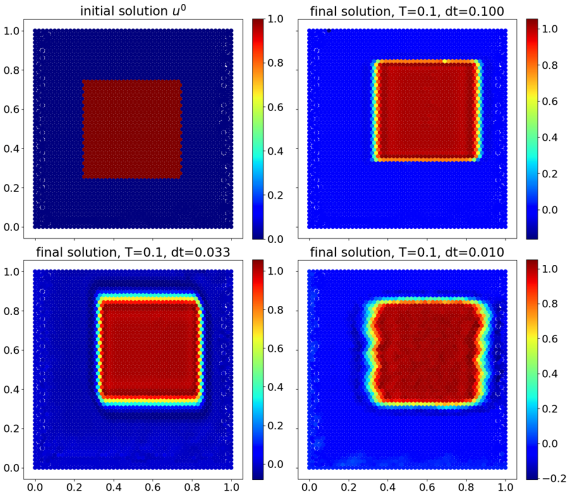

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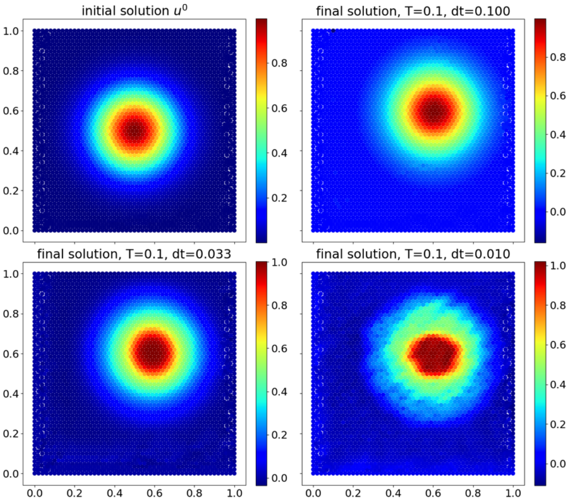

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