University of Strasbourg

Master 1 CSMI

Active swimming

Lucas Jacquin

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Contents

1 Introduction

Elasticity is the property of a material to change its shape during the application of a force (external and/or internal), and to regain its original shape afterwards.

While the industrial applications of elasticity (most notably elasticity in construction and engineering materials) are numerous, we're more interested in the medical applications.

2 Passive Elasticity

Passive elasticity is the study of the deformation of a material under exclusively external forces. If a body $\Omega \subset \mathbb{R}^d$ of density ρ is subjected to an external body force f, the equations of passive elasticity on Ω in terms of the displacement η are given by:

$$\begin{aligned} \rho \frac{\partial^2 \eta}{\partial t^2} - \nabla \cdot (F\Sigma) = f \text{ in } \Omega \ , \\ \eta = g_D \quad \text{on} \quad \Gamma_D \ , \\ F\Sigma \mathbf{n} = g_N \quad \text{on} \quad \Gamma_N \ , \end{aligned} \tag{1}$$

where $F = \mathbf{I} + \nabla\eta$ is the deformation gradient, $\mathbf{I}$ the identity matrix of $\mathbb{R}^d$, and Σ is the second Piola-Kirchoff stress tensor which describes the passive elastic behavior of the structure. In the Saint Venant–Kirchhoff model, the second Piola-Kirchoff stress tensor is

$$\Sigma = \lambda \text{tr}\,(E)\,\mathbf{I} + 2\mu E, \quad E = \frac{1}{2}\left(\nabla\eta + \nabla\eta^T + \nabla\eta^T\nabla\eta\right) \ , \tag{2}$$

where λ and μ are the Lamé coefficients

$$\lambda = \frac{E\nu}{(1-\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)} \ ,$$

expressed in terms of Young's modulus E and Poisson's ratio ν which represent respectively the stiffness of the medium and its compressibility.

3 Active Elasticity

Active elasticity is the study of the deformation of a material under both internal and external forces.

3.1 Active-Stress

The active-stress point of view assumes that the internal at any given point during the displacement, elastic deformations are made in a single direction, called active fiber direction. This model is very well suited to simulate active

elasticity on a body that, at the macroscopic scale (when we average the microscopic active components), exhibits a fiber-like structure.
The active fiber direction is denoted by $e_a(x,t)$, but in none of the examples we study will $e_a(x,t)$ depend on time or position, and so we'll denote it by e_a.

We introduce an active stress tensor Σ_*, defined by

$$\Sigma^* = \Sigma_a e_a \otimes e_a,$$

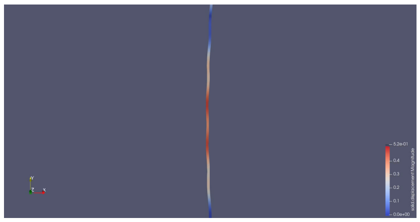
where Σ_a is a scalar function describing the stretching-elongating behavior of the active fibers which also depends on the time and the material position, and $\otimes$ denotes the tensor product.
The active-stress point of view then consists in modifying the passive elasticity equations (1) by changing the second Piola-Kirchoff stress tensor Σ in $\Sigma - \Sigma^*$:

$$\begin{aligned} \rho\frac{\partial^2\eta}{\partial t^2} - \nabla \cdot (F(\Sigma - \Sigma^*)) = f \text{ in } \Omega \ , \\ \eta = g_D \quad \text{on} \quad \Gamma_D \ , \\ (F\Sigma - F\Sigma^*)\mathbf{n} = g_N \quad \text{on} \quad \Gamma_N \ . \end{aligned} \tag{3}$$

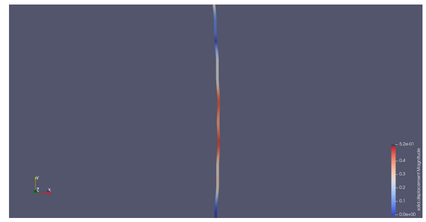
We implemented a Finite Element Method in Feel++ using the Computational Solid Mechanics Toolbox to solve the active stress problem. Using the algebraic factory tools and following the Newton linearization process, we decomposed the non-linear terms using a Taylor development in the variational formulation in a linear part, a jacobian-dependent part and a residual part.

The implemented code can be found in `solid_active_additive.cpp`.

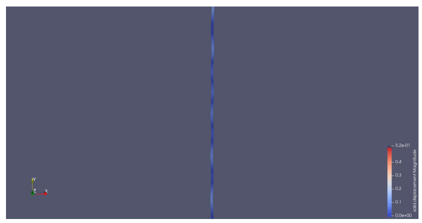
Here are some of the results we obtained in modeling the movements of a 2D pulmonary cilium, using the active stress FEM.



(a) t = 0.01

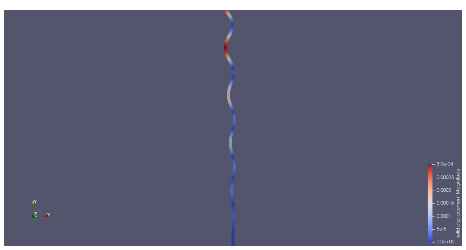


(b) t = 0.03



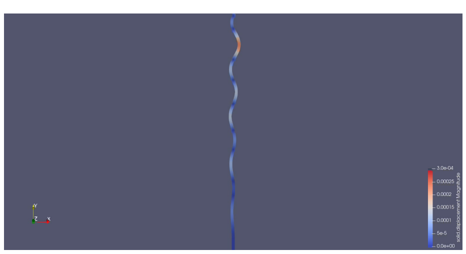
(c) t = 0.02

Figure 1: Numerical simulation for flapping cilia with at unit scale

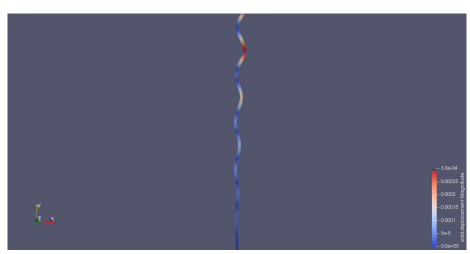


(a) $t = 0.075$

(b) $t = 0.1$



(c) $t = 0.125$



(d) $t = 0.180$

Figure 2: Numerical simulation for flapping cilia with varying amplitude

3.2 Active-Strain

We started by constructing a mesh that we could use our active-strain elasticity model on, based on the work by Curatolo and Teresi in [1]. The idea was to have a material with several layers, each having different Young moduli; and that would mimic the flapping of a fish.

We began by focusing on the *several layers* part, with an approximation of a fish built as follows:

- Five 1×10 rectangular layers stacked vertically in a rectangle
- The outer layers have the same Young modulus
- The first inner layers have the same Young modulus, higher than the outer layer's
- The innermost layer has the highest Young modulus

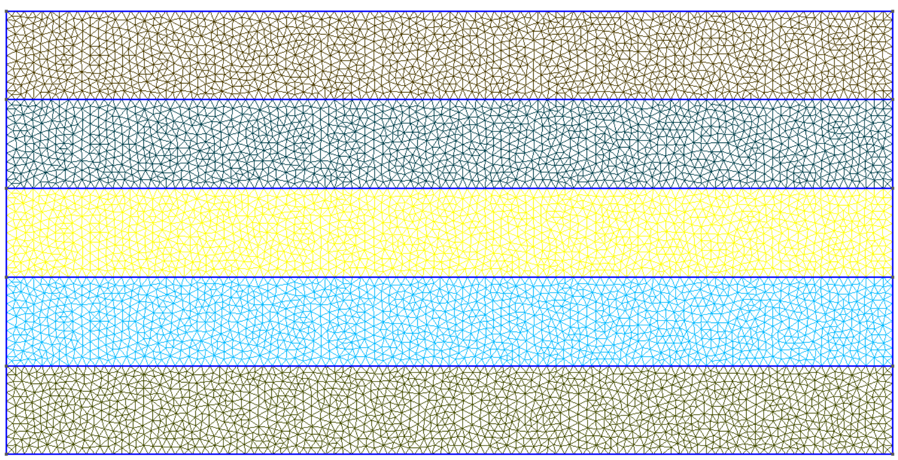


Figure 3: Sandwich-like approximation

The `gmsh` file of this mesh can be found in the `sandwich` folder, along with `json` and `cfg` files to use with the Feel++ toolbox plugin `feelpp_p_multiplicative`.

We then built a better approximation of a fish:

We denote by L the fish length and h it's width, the two extreme horizontal points coordinates are given by $(0, 0)$ and $(L, 0)$.
Using the polynomial coefficients given in [1], we can get the Y-coordinates (dependent on h) of the border points for $X = 0.25L$, $X = 0.5L$ and $X = 0.75L$.
We use more interpolation points for the head so that it more closely resembles a fish's ($X = 0.85L$, $X = 0.9L$ and $X = 0.95L$), and use the `spline` functionality of `gmsh` to get an interpolation curve of the points we built.

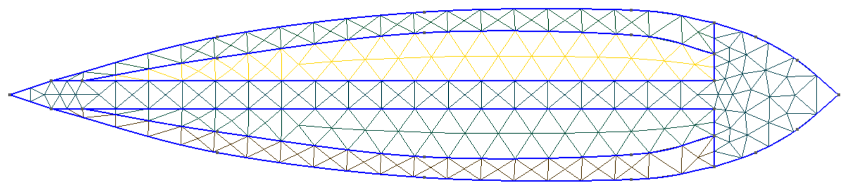


Figure 4: Fish mesh

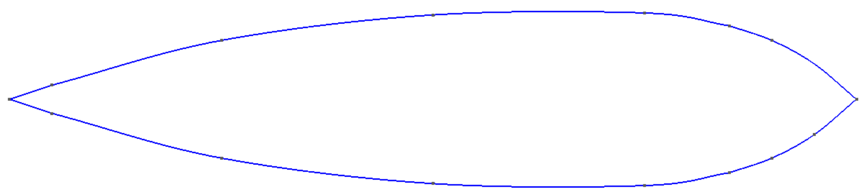


Figure 5: Fish outline

We then implemented a program to solve the FEM associated with the active-strain problem, using only the Feel++ library at first. The code doesn't work properly on our fish model: at certain time steps, the linear solver fails to converge. We did get some interesting results nonetheless:

Eventually, we tried to adapt this code using the CSM Toolbox, but unfortunately we didn't manage to make it work.

References

[1] Teresi Curatolo. Modeling and simulation of fish swimming with active muscles. 2016.