Elasticity and toolbox Coefficient Form PDEs

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1 Introduction

We want to express the elasticity equations in the generic form of the toolbox Coefficient Form PDEs given by:

$$d\frac{\partial u}{\partial t} + \nabla \cdot (-c\nabla u - \alpha u + \gamma) + \beta \cdot \nabla u + au = f \tag{1}$$

2 Stationary linear elasticity in dimension 2

We solve the following problem :

$$-\nabla \cdot \sigma(\eta) = f \tag{2}$$

where :

$$\begin{array}{cccc} \eta & : & S & \rightarrow & \mathbf{R}^2 \\ & & \eta(X) & \mapsto & (\eta_1(X), \eta_2(X)), X \in S \subseteq \mathbf{R}^2 \end{array}$$

In order to use the toolbox, the equation (2) must be written component by component. To do this, we start by calculating $\sigma(\eta)$:

$$\sigma(\eta) = \lambda tr(e(\eta))I_2 + 2\mu e(\eta) \tag{3}$$

We have :

$$\begin{aligned}
&\nabla\eta = \begin{pmatrix} \partial_x\eta_1 & \partial_y\eta_1 \\ \partial_x\eta_2 & \partial_y\eta_2 \end{pmatrix}, \nabla\eta^T = \begin{pmatrix} \partial_x\eta_1 & \partial_x\eta_2 \\ \partial_y\eta_1 & \partial_y\eta_2 \end{pmatrix} \\
&e(\eta) = \frac{1}{2}(\nabla\eta + \nabla\eta^T) = \frac{1}{2}\begin{pmatrix} 2\partial_x\eta_1 & \partial_y\eta_1 + \partial_x\eta_2 \\ \partial_x\eta_2 + \partial_y\eta_1 & 2\partial_y\eta_2 \end{pmatrix} \\
&tr(e(\eta)) = \nabla \cdot \eta
\end{aligned}$$

So, we get for (3) :

$$\sigma(\eta) = \lambda \begin{pmatrix} \nabla \cdot \eta & 0 \\ 0 & \nabla \cdot \eta \end{pmatrix} + \mu \begin{pmatrix} 2\partial_x\eta_1 & \partial_y\eta_1 + \partial_x\eta_2 \\ \partial_x\eta_2 + \partial_y\eta_1 & 2\partial_y\eta_2 \end{pmatrix}$$

We want to solve the system (2) component by component. This new problem is given by :

$$\begin{cases} -\nabla \cdot \sigma_{1,:} & = f_1 \\ -\nabla \cdot \sigma_{2,:} & = f_2 \end{cases} \tag{4}$$

Using our expression of $\sigma(\eta)$, we get for the first equation of (4):

$$\begin{aligned}
-\nabla \cdot \sigma_{1,:} &= -\nabla \cdot (\lambda[\nabla \cdot \eta, 0] + \mu[2\partial_x\eta_1, \partial_y\eta_1 + \partial_x\eta_2]) \\
&= -\nabla \cdot (\lambda[\nabla \cdot \eta, 0] + \mu[\partial_x\eta_1, \partial_y\eta_1] + \mu[\partial_x\eta_1, \partial_x\eta_2]) \\
&= -\nabla \cdot (\mu\nabla\eta_1 + \lambda[\partial_x\eta_1 + \partial_y\eta_2, 0] + \mu[\partial_x\eta_1, \partial_x\eta_2]) \\
&= -\nabla \cdot (\mu\nabla\eta_1 + [(\lambda + \mu)\partial_x\eta_1 + \lambda\partial_y\eta_2, \mu\partial_x\eta_2]) \\
&= \nabla \cdot (-\mu\nabla\eta_1 - [(\lambda + \mu)\partial_x\eta_1 + \lambda\partial_y\eta_2, \mu\partial_x\eta_2])
\end{aligned}$$

So to write this equation in generic form (1), we have to choose :

$u = \eta_1$, $c = \mu$, $\gamma = -[(\lambda + \mu)\partial_x\eta_1 + \lambda\partial_y\eta_2, \mu\partial_x\eta_2]$ and $f = f_1$.

All the other coefficients are equal to 0.

For the second equation of (4) we do the same calculations :

$$\begin{aligned}
-\nabla \cdot \sigma_{2,:} &= -\nabla \cdot (\lambda[0, \nabla \cdot \eta] + \mu[\partial_x\eta_2 + \partial_y\eta_1, 2\partial_y\eta_2]) \\
&= -\nabla \cdot (\lambda[0, \nabla \cdot \eta] + \mu[\partial_x\eta_2, \partial_y\eta_2] + \mu[\partial_y\eta_1, \partial_y\eta_2]) \\
&= -\nabla \cdot (\mu\nabla\eta_2 + \lambda[0, \partial_x\eta_1 + \partial_y\eta_2] + \mu[\partial_y\eta_1, \partial_y\eta_2]) \\
&= -\nabla \cdot (\mu\nabla\eta_2 + [\mu\partial_y\eta_1, (\lambda + \mu)\partial_y\eta_2 + \lambda\partial_x\eta_1]) \\
&= \nabla \cdot (-\mu\nabla\eta_2 - [\mu\partial_y\eta_1, (\lambda + \mu)\partial_y\eta_2 + \lambda\partial_x\eta_1])
\end{aligned}$$

In generic form (1), we have to choose :

$u = \eta_2$, $c = \mu$, $\gamma = -[\mu\partial_y\eta_1, (\lambda + \mu)\partial_y\eta_2 + \lambda\partial_x\eta_1]$ and $f = f_2$.

All the other coefficients are equal to 0.

For the implementation of the stationary linear elasticity (2) in the toolbox cfpdes, we write the two equations of (4) with the coefficients defined as above.

We can then visualize η using Paraview, by exporting the vectorial expression $[\eta_1, \eta_2]$:

```
"eta":
{
    "expr":"{equation1_eta1,equation2_eta2}:equation1_eta1:equation2_eta2"
}
```

2.1 Results

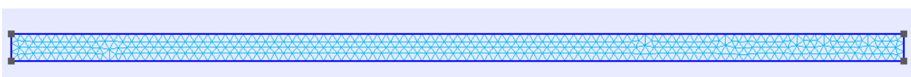
For the tests, we use a material for which the Poisson ration ν and the Young modulu E are set to :

$$\nu = 0.3, E = 1e6$$

So, the Lamé coefficients are equal to :

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \mu = \frac{E}{2(1+\nu)}$$

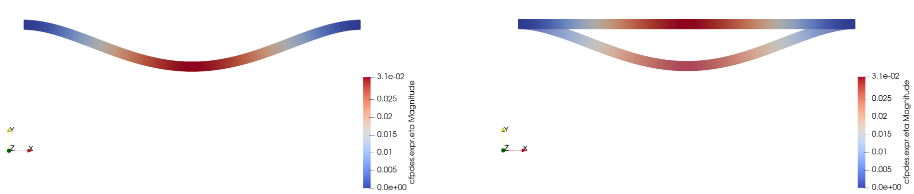
For the simulations, we use a beam. With a refinement $h = 0.1$, we have the following mesh:



First test: beam clamped at two sides subject to gravity

In this case, we have homogeneous Dirichlet conditions on the left and right side of the beam. We choose for the mesh a refinement $h = 0.01$. As the beam is subjected to gravity, $f = (0, -9.81)$.

Visualization of η obtained with Paraview :



(a) Scale factor = 40

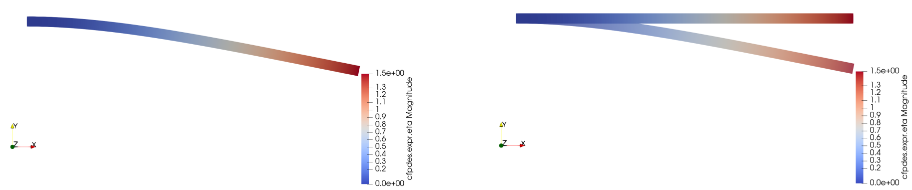
(b) Scale factor = 0, 40

Figure 1: Beam clamped at two sides subject to gravity

Second test: beam clamped on one side subject to gravity

We have homogeneous Dirichlet conditions on the left side of the beam. We use $h = 0.01$ for the mesh. The beam is subjected to gravity, so $f = (0, -9.81)$.

Visualization of η obtained with Paraview :



(a) Scale factor = 1

(b) Scale factor = 0, 1

Figure 2: Beam clamped on one side subject to gravity

3 Linear elasticity in dimension 2

Now we are interested in the equation :

$$\partial_{tt}\eta - \nabla \cdot \sigma(\eta) = f \tag{5}$$

where :

$$\begin{array}{cccc} \eta & : & [0,T]\text{x}S & \to & \mathbf{R}^2 \\ & & \eta(t,X) & \mapsto & (\eta_1(t,X), \eta_2(t,X)), X \in S \subseteq \mathbf{R}^2 \end{array}$$

Since we cannot solve second time derivatives with this toolbox, we must first calculate the first time derivative. We obtain the following system of equations:

$$\begin{cases} \partial_t \eta_1 = v_1 \\ \partial_t \eta_2 = v_2 \\ \partial_t v_1 - \nabla \cdot \sigma_{1,:} = f_1 \\ \partial_t v_2 - \nabla \cdot \sigma_{2,:} = f_2 \end{cases} \tag{6}$$

We have to implement this four equations. Let's check how to write them in the generic form (1).

For the first equation of (6), we have directly :

$$u = \eta_1, d = 1, f = v_1$$

All other coefficients are equal to 0.

In the same way, we have for the second equation :

$$u = \eta_2, d = 1, f = v_2$$

Consider now the third equation of (6). As we have a time derivative of v_1, this variable represents our unknown. As $\nabla \cdot \sigma_{1,:}$:

$$-\nabla \cdot \sigma_{1,:} = \nabla \cdot (-\mu \nabla \eta_1 - [(\lambda + \mu)\partial_x \eta_1 + \lambda \partial_y \eta_2, \mu \partial_x \eta_2])$$

does not depend on v_1 we have to rewrite this expression and put it in γ.

$$\begin{aligned} -\nabla \cdot \sigma_{1,:} &= \nabla \cdot (-\mu \nabla \eta_1 - [(\lambda + \mu)\partial_x \eta_1 + \lambda \partial_y \eta_2, \mu \partial_x \eta_2]) \\ &= \nabla \cdot (-\mu [\partial_x \eta_1, \partial_y \eta_1] - [(\lambda + \mu)\partial_x \eta_1 + \lambda \partial_y \eta_2, \mu \partial_x \eta_2]) \\ &= \nabla \cdot (-[(\lambda + 2\mu)\partial_x \eta_1 + \lambda \partial_y \eta_2, \mu \partial_y \eta_1 + \mu \partial_x \eta_2]) \end{aligned}$$

So for this equation we find :

$$u = v_1, d = 1, \gamma = -[(\lambda + 2\mu)\partial_x \eta_1 + \lambda \partial_y \eta_2, \mu \partial_y \eta_1 + \mu \partial_x \eta_2], f = f_1$$

For the last equation, we do the same calculations:

$$\begin{aligned} -\nabla \cdot \sigma_{2,:} &= \nabla \cdot (-\mu \nabla \eta_2 - [\mu \partial_y \eta_1, (\lambda + \mu)\partial_y \eta_2 + \lambda \partial_x \eta_1]) \\ &= \nabla \cdot (-[\mu \partial_x \eta_2 + \mu \partial_y \eta_1, (\lambda + 2\mu)\partial_y \eta_2 + \lambda \partial_x \eta_1]) \end{aligned}$$

So we have for the fourth equation of (6):

$$u = v_2, d = 1, \gamma = -[\mu \partial_x \eta_2 + \mu \partial_y \eta_1, (\lambda + 2\mu)\partial_y \eta_2 + \lambda \partial_x \eta_1], f = f_2$$

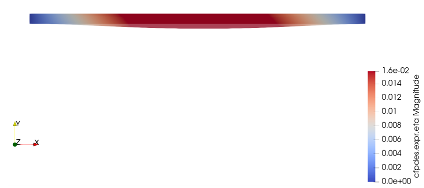
3.1 Results

We do the same tests as for the stationary linear elasticity. For the four unknowns, η_1, η_2, v_1, v_2, we consider initial conditions equal to 0. We do the simulations on the time interval $[0s, 3s]$, with a step of $0.1s$.

First test: beam clamped at two sides subject to gravity

The visualization of η obtained with Paraview allows to observe the movement of the beam. To better visualize this movement, we use a scale factor equal to 10.

(a) t = 0.0



(b) t = 0.4

(c) t = 0.6

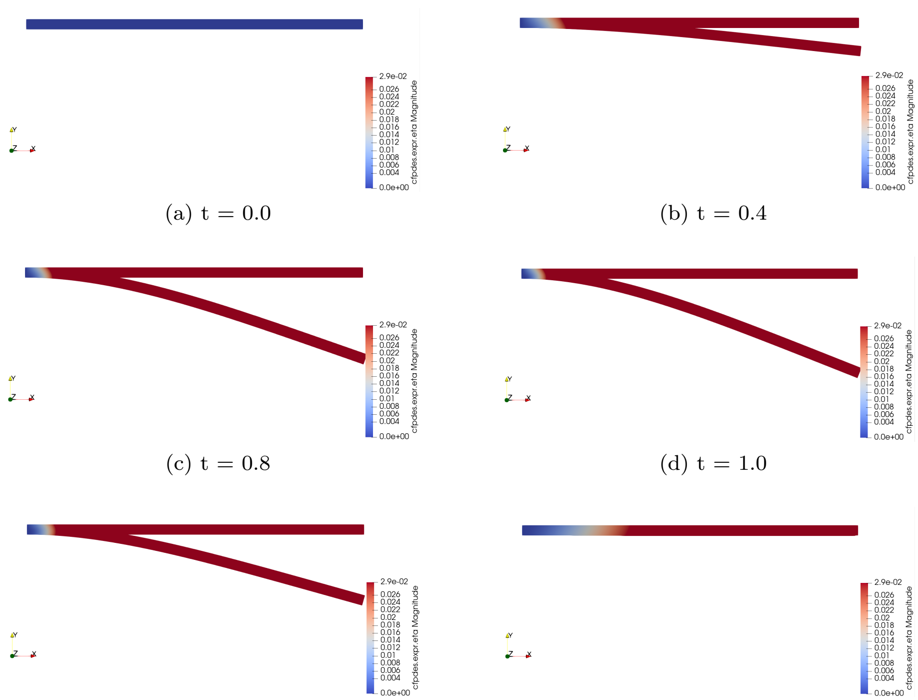
(d) t = 0.8

(e) t = 1.0

Figure 3: Beam clamped at two sides subject to gravity

Second test: beam clamped on one side subject to gravity

Depending on the time, we obtain, with a scale factor equal to 1, the following visualization of η :



(a) t = 0.0 (b) t = 0.4

(c) t = 0.8 (d) t = 1.0

(e) t = 1.4 (f) t = 2.0

Figure 4: Beam clamped on one side subject to gravity

4 Additive model for active elasticity

We consider the following problem :

$$\begin{cases} \partial_{tt}\eta - \nabla \cdot [\sigma - F\Sigma^*] = f & \text{in S} \\ \sigma n = F\Sigma^* n & \text{on } \partial S_N \\ \eta = 0 & \text{on } \partial S_D \end{cases} \tag{7}$$

where :

- $F(t, X) = I_2 + \nabla\eta$, the deformation gradient.
- $\Sigma^* = \Sigma_a e_a \otimes e_a$. the active elasticity stress tensor.
- e_a a vector representing the direction of active fibres.
- Σ_a a scalar function.

We like to rewrite the problem (7) in the generic form of the toolbox cfpdes (1). We write the equations component by component and we obtain a new problem :

$$\begin{cases} \partial_t\eta_1 = v_1 & \text{in S} \\ \partial_t\eta_2 = v_2 & \text{in S} \\ \partial_t v_1 - \nabla \cdot [\sigma_{1,:} - (F\Sigma^*)_{1,:}] = f_1 & \text{in S} \\ \partial_t v_2 - \nabla \cdot [\sigma_{2,:} - (F\Sigma^*)_{2,:}] = f_2 & \text{in S} \\ (\sigma_{1,:} - (F\Sigma^*)_{1,:})n = 0 & \text{on } \partial S_N \\ (\sigma_{2,:} - (F\Sigma^*)_{2,:})n = 0 & \text{on } \partial S_N \\ \eta_1 = 0 & \text{on } \partial S_D \\ \eta_2 = 0 & \text{on } \partial S_D \end{cases} \tag{8}$$

The first two equations of this system are equal to those of the linear elasticity case. To define the next two, we start by analyzing the term : $F\Sigma^*$. We have :

$$\Sigma^* = \Sigma_a e_a \otimes e_a$$

If we assume that $e_a = [e_{a1}, e_{a2}]$, we get :

$$\Sigma^* = \Sigma_a \begin{pmatrix} e_{a1}e_{a1} & e_{a1}e_{a2} \\ e_{a1}e_{a2} & e_{a2}e_{a2} \end{pmatrix} = \begin{pmatrix} \Sigma_a e_{a1}e_{a1} & \Sigma_a e_{a1}e_{a2} \\ \Sigma_a e_{a1}e_{a2} & \Sigma_a e_{a2}e_{a2} \end{pmatrix}$$

So :

$$\begin{aligned} F\Sigma^* &= (I_2 + \nabla\eta) \begin{pmatrix} \Sigma_a e_{a1}e_{a1} & \Sigma_a e_{a1}e_{a2} \\ \Sigma_a e_{a1}e_{a2} & \Sigma_a e_{a2}e_{a2} \end{pmatrix} \\ &= \begin{pmatrix} 1 + \partial_x\eta_1 & \partial_y\eta_1 \\ \partial_x\eta_2 & 1 + \partial_y\eta_2 \end{pmatrix} \begin{pmatrix} \Sigma_a e_{a1}e_{a1} & \Sigma_a e_{a1}e_{a2} \\ \Sigma_a e_{a1}e_{a2} & \Sigma_a e_{a2}e_{a2} \end{pmatrix} \\ &= \begin{pmatrix} (1 + \partial_x\eta_1)(\Sigma_a e_{a1}e_{a1}) + (\partial_y\eta_1)(\Sigma_a e_{a1}e_{a2}) & (1 + \partial_x\eta_1)(\Sigma_a e_{a1}e_{a2}) + (\partial_y\eta_1)(\Sigma_a e_{a2}e_{a2}) \\ (\partial_x\eta_2)(\Sigma_a e_{a1}e_{a1}) + (1 + \partial_y\eta_2)(\Sigma_a e_{a1}e_{a2}) & (\partial_x\eta_2)(\Sigma_a e_{a1}e_{a2}) + (1 + \partial_y\eta_2)(\Sigma_a e_{a2}e_{a2}) \end{pmatrix} \end{aligned}$$

This calculation allows us to deduce that :

$$(F\Sigma^*)_{1,:} = [(1+\partial_x\eta_1)(\Sigma_a e_{a1}e_{a1}) + (\partial_y\eta_1)(\Sigma_a e_{a1}e_{a2}), (1+\partial_x\eta_1)(\Sigma_a e_{a1}e_{a2}) + (\partial_y\eta_1)(\Sigma_a e_{a2}e_{a2})]$$
$$(F\Sigma^*)_{2,:} = [(\partial_x\eta_2)(\Sigma_a e_{a1}e_{a1}) + (1+\partial_y\eta_2)(\Sigma_a e_{a1}e_{a2}), (\partial_x\eta_2)(\Sigma_a e_{a1}e_{a2}) + (1+\partial_y\eta_2)(\Sigma_a e_{a2}e_{a2})]$$

Let us now consider the third equation of the problem (8). In the generic form (1), the unknown is v_1. So, the term $-(\sigma_{1,:} - (F\Sigma^*)_{1,:})$ must be put in γ, with $\sigma_{1,:}$ defined as in the part of the linear elasticity. We do same for the fourth equation of (8) with $\gamma = -(\sigma_{2,:} - (F\Sigma^*)_{2,:})$.

4.1 Internal activity for a pulmonary cilium

In this case, we consider $e_a = [0,1]$ We can rewrite our expressions for $F\Sigma^*$:

$$(F\Sigma^*)_{1,:} = [0, (\partial_y\eta_1)\Sigma_a]$$
$$(F\Sigma^*)_{2,:} = [0, (1+\partial_y\eta_2)\Sigma_a]$$

With these values for the vector e_a the γ expression for the third and fourth equation of (8) becomes easier :

$$-(\sigma_{1,:} - (F\Sigma^*)_{1,:}) = -[(\lambda+2\mu)\partial_x\eta_1 + \lambda\partial_y\eta_2, \mu\partial_y\eta_1 + \mu\partial_x\eta_2 - (\partial_y\eta_1)\Sigma_a]$$
$$-(\sigma_{2,:} - (F\Sigma^*)_{2,:}) = -[\mu\partial_x\eta_2 + \mu\partial_y\eta_1, (\lambda+2\mu)\partial_y\eta_2 + \lambda\partial_x\eta_1 - (1+\partial_y\eta_2)\Sigma_a]$$

4.1.1 Bending

4.1.2 Flapping